Midterm Exam 3

Math 153H, Spring 2008

You have 50 minutes. No notes, no books, no calculators. Good luck!

Name: Solutions

ID #: ________________

1. ________ (/20 points)

2. ________ (/20 points)

3. ________ (/20 points)

4. ________ (/20 points)

5. ________ (/20 points)

Total ________ (/100 points)

Homework Average ________

Course Average ________

1. [20 points] Evaluate the following integrals.

(a) $\int_0^1 \frac{x}{\sqrt{1-x^2}}\,dx \quad = \int_0^{\pi/2} \frac{\sin\theta}{\cos\theta}\cdot\cos\theta\,d\theta$

$x = \sin\theta$

$dx = \cos\theta\,d\theta$

$\sqrt{1-x^2} = \cos\theta$

$= \int_0^{\pi/2} \sin\theta\,d\theta$

$= -\cos\theta\,\Big]_0^{\pi/2}$

$= 0 - (-1)$

$= 1.$

(b) $\int \frac{8x-10}{(x+1)(x-1)(x-2)}\,dx$

$$\frac{8x-10}{(x+1)(x-1)(x-2)} = \frac{A}{x+1} + \frac{B}{x-1} + \frac{C}{x-2}$$

$$8x-10 = A(x-1)(x-2) + B(x+1)(x-2) + C(x+1)(x-1)$$

$x = -1 \Rightarrow 8(-1)-10 = A(-2)(-3) \qquad -18 = 6A \qquad \boxed{A=-3}$

$x = 1 \Rightarrow 8(1)-10 = B(2)(-1) \qquad -2 = -2B \qquad \boxed{B = 1}$

$x = 2 \Rightarrow 8(2)-10 = C(3)(1) \qquad 6 = 3C \qquad \boxed{C = 2}$

$$\int \frac{8x-10}{(x+1)(x-1)(x-2)} = \int \frac{-3}{x+1} + \frac{1}{x-1} + \frac{2}{x-2}\,dx$$

$$= -3\ln|x+1| + \ln|x-1| + 2\ln|x-2| + C.$$

2. [20 points] Do the following improper integrals converge or diverge? Justify your answer.

(a) $\int_0^\infty x\, e^{-x}\, dx$ $= -x e^{-x}\Big]_0^\infty + \int_0^\infty e^{-x}\, dx$

Integrate by parts.
$u = x \quad dv = e^{-x} dx$
$du = dx \quad v = -e^{-x}$

$= \left[-x e^{-x} - e^{-x}\right]_0^\infty$

$= \lim_{b\to\infty} (-b-1)e^{-b} - 0 - (-1)$

$= \lim_{b\to\infty} \frac{-b-1}{e^b} + 1$

L'H
$= \lim_{b\to\infty} \frac{-1}{e^b} + 1$

$= 1.$

So the integral converges.

(b) $\int_0^\infty \frac{3+\cos x}{x}\, dx$ $\geq \int_0^\infty \frac{2}{x}\, dx$, which diverges (both as $x \to 0$ and $x \to \infty$).

$2 \leq 3 + \cos x \leq 4$

Thus, by the comparison test, $\int_0^\infty \frac{3+\cos x}{x}\, dx$ diverges.

3. [20 points] Compute the following limits.

(a) $\lim_{x\to 0^+} (\sin x)^x \quad = L.$

$$\lim_{x\to 0^+} x \ln(\sin x) = \ln L.$$

$$= \lim_{x\to 0^+} \frac{\ln(\sin x)}{1/x}$$

$$\overset{L'H}{=} \lim_{x\to 0^+} \frac{(\cos x)/(\sin x)}{-1/x^2}$$

$$= \lim_{x\to 0^+} \frac{-x^2 \cos x}{\sin x}$$

$$\overset{L'H}{=} \lim_{x\to 0^+} \frac{x^2 \sin x - 2x \cos x}{\cos x}$$

$$= 0.$$

since $\ln L = 0$,

$\underline{L = 1.}$

(b) $\lim_{n\to\infty} \frac{1}{\sqrt{n+\sqrt{n}}-\sqrt{n}} \cdot \frac{\sqrt{n+\sqrt{n}}+\sqrt{n}}{\sqrt{n+\sqrt{n}}+\sqrt{n}}$

$$= \lim_{n\to\infty} \frac{\sqrt{n+\sqrt{n}}+\sqrt{n}}{(n+\sqrt{n})-n}$$

$$= \lim_{n\to\infty} \frac{\sqrt{n+\sqrt{n}}+\sqrt{n}}{\sqrt{n}}$$

$$= \lim_{n\to\infty} \sqrt{1+\frac{1}{\sqrt{n}}} + 1$$

$$= 2.$$

4. [20 points] Consider the function $f(x) = \sin(x^2)$ on the interval $[0, \pi]$.

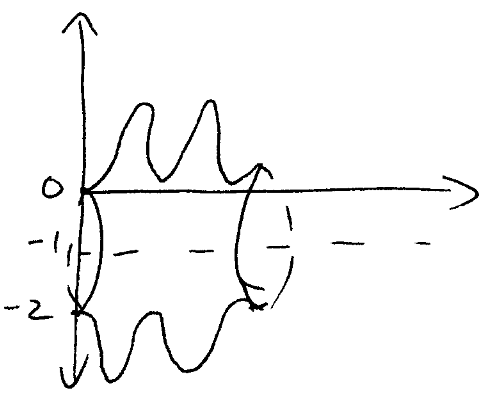
(a) Set up an integral that expresses the length of the curve $y = f(x)$. *You do not have to evaluate the integral!*

$$f'(x) = 2x\cos(x^2)$$

$$\text{So}\quad ds = \sqrt{1 + (f'(x))^2}\,dx = \sqrt{1 + 4x^2\cos^2(x^2)}\,dx$$

$$\text{Arclength} = \int_0^{\pi} \sqrt{1 + 4x^2\cos^2(x^2)}\,dx$$

(b) Suppose the graph of $f(x)$ is revolved around the line $y = -1$. Set up an integral that expresses the area of the resulting surface. *Again, you do not have to evaluate the integral!*



$$\text{Area} = \int_0^{\pi} 2\pi r \cdot ds$$

$$= \int_0^{\pi} 2\pi\left(\sin(x^2) + 1\right)\sqrt{1 + 4x^2\cos^2(x^2)}\,d$$

5. [20 points] Suppose you are trying to find the root of $y = \sqrt[3]{x}$. Describe what will happen when you search for the root with Newton's method, starting from an initial guess of $x_0 = 1$.

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$= x_n - \frac{(x_n)^{1/3}}{\frac{1}{3}(x_n)^{-2/3}}$$

$$= x_n - 3x_n$$

$$= -2x_n.$$

So, if $x_0 = 1$, we have

$x_1 = -2$

$x_2 = 4$

$x_3 = -8$

$x_4 = 16$

etc.

Thus Newton's method takes us further and further away from the root at $x = 0$.