Midterm Exam 2

Math 153H, Spring 2008

You have 50 minutes. No notes, no books, no calculators. Good luck!

Name: Solutions

ID #: __________

1. ________ (/20 points)

2. ________ (/20 points)

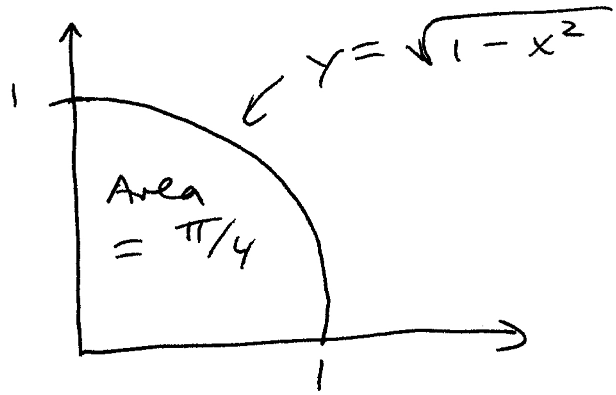
3. ________ (/20 points)

4. ________ (/20 points)

5. ________ (/20 points)

Total ________ (/100 points)

1. [20 points] Let R be the region in the plane bounded by the unit circle, the positive x–axis, and the positive y–axis. In other words, R is the part of the unit disk that's contained in the first quadrant. Find the center of mass of R.



Center of mass: $\boxed{\left(\frac{4}{3\pi}, \frac{4}{3\pi}\right).}$

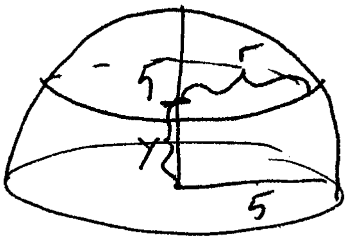
$$\bar{x} = \frac{\int_0^1 x\sqrt{1-x^2}\,dx}{\pi/4} \qquad u = 1-x^2, \quad du = -2x\,dx$$

$$\bar{x} = \frac{\int_1^0 -\frac{1}{2} u^{1/2}\,du}{\pi/4} = \frac{\frac{1}{2}\int_0^1 u^{1/2}\,du}{\pi/4} = \frac{\frac{1}{2}\cdot\frac{2}{3}u^{3/2}\Big]_0^1}{\pi/4} = \frac{4}{\pi}\cdot\frac{1}{2}\cdot\frac{2}{3} = \frac{4}{3\pi}.$$

$$\bar{y} = \frac{\int_0^1 \frac{1}{2}(1-x^2)\,dx}{\pi/4} = \frac{\frac{1}{2}\left[x - \frac{x^3}{3}\right]_0^1}{\pi/4} = \frac{1}{2}\left(1-\frac{1}{3}\right)\cdot\frac{4}{\pi} = \frac{1}{2}\cdot\frac{2}{3}\cdot\frac{4}{\pi} = \frac{4}{3\pi}.$$

(Note that by symmetry, we expect $\bar{x} = \bar{y}$.)

2. [20 points] A planned skate park is supposed to have a concrete mound in the shape of a solid hemisphere whose radius is $5\,m$. Given that the density of concrete is $4000\,kg/m^3$, compute the amount of work required to lift all the concrete from ground level into its proper position. (You may use the value of $10\,m/s^2$ for the acceleration due to gravity.)



$r^2 + y^2 = 25$

$r^2 = 25 - y^2$

Volume of slice : $\pi r^2 \Delta y = \pi(25 - y^2)\Delta y$ (m^3)

Mass of slice : $4000\pi(25 - y^2)\Delta y$

$= (10^5\pi - 4\times10^3\pi y^2)\Delta y$ (kg)

Force acting on slice :

$10^6\pi - 4\times10^4\pi y^2 \Delta y$ N.

Distance raised : y.

$$\text{Work} = \int_0^5 10^6\pi y - 4\times10^4\pi y^3\,dy$$

$$= \frac{1}{2}\times10^6\pi y^2\Big]_0^5 - 10^4\pi y^4\Big]_0^5$$

$$= \frac{25}{2}\times10^6\pi - 625\times10^4\pi$$

$$= 1.25\times10^7\pi - 6.25\times10^6\pi$$

$$= 6.25\times10^6\pi \text{ Joules.}$$

Estimate : Volume of mound is $\frac{1}{2}\cdot\frac{4}{3}\pi r^3 = \frac{2}{3}\pi\cdot5^3$

$= \frac{250}{3}\pi\ m^3$

Mass : $\frac{10^6}{3}\pi$ kg. Force : $\frac{10^7}{3}$ N.

Distance moved is about 2 m.

So estimate is $\frac{2}{3}\times10^7 \approx 6.7\times10^6$ J. ✓

3. [20 points] Solve the differential equation $\frac{dy}{dx} = \frac{-y}{x} + \sin(x)$, where $y(\pi) = 0$.

$\frac{dy}{dx} = P(x) \cdot y + Q(x)$, $P(x) = \frac{-1}{x}$, $Q(x) = \sin x$

$\int P(x)\,dx = -\ln|x| + C.$

Integrating factor: $e^{-\int P(x)\,dx} = e^{\ln|x|} = |x|$

$= x$ when $x \geq 0$.

So we rewrite the equation as

$\frac{dy}{dx} + \frac{y}{x} = \sin x$

$u = x$ $\quad dv = \sin x\,dx$

$du = dx$ $\quad v = -\cos x$

$x\frac{dy}{dx} + y = x \sin x$

$\frac{d}{dx}(xy) = x \sin x.$

$xy = \int x \sin x\,dx$

$xy = -x\cos x + \int \cos x\,dx$

$xy = -x\cos x + \sin x + C$

$y = -\cos x + \frac{\sin x}{x} + \frac{C}{x}.$

when $x = \pi$, $y = -\cos(\pi) + \frac{\sin(\pi)}{\pi} + \frac{C}{\pi} = 0.$

$1 + 0 + \frac{C}{\pi} = 0.$

$C/\pi = -1$ $\quad C = -\pi$

So $\boxed{y = -\cos x + \frac{\sin x}{x} - \frac{\pi}{x}.}$ (for $x > 0$)

4. [20 points] A rumor is spreading through the population of a dorm. The rate of spread of the rumor is proportional to the number of people who know it and the number of people who don't know it. In other words, if t is time in hours and $P(t)$ is the proportion of the student body that knows the rumor,

$$\frac{dP}{dt} = \ln(2)\, P(1-P).$$

Given that $1/5$ of the students know the rumor at time $t = 0$, what fraction of the students will know the rumor 3 hours later?

Hint: You can use the identity $\displaystyle \frac{1}{P(1-P)} = \frac{1}{P} + \frac{1}{1-P}$.

$$\frac{dP}{P(1-P)} = \ln(2)\, dt$$

$$\frac{dP}{P} + \frac{dP}{1-P} = \ln(2)\, dt$$

$$\ln|P| - \ln|1-P| = \ln(2)\, t + C$$

$$\ln\left|\frac{P}{1-P}\right| = \ln(2)\, t + C$$

$$\frac{P}{1-P} = Ae^{\ln(2)\, t} = A\cdot 2^t$$

when $t=0$, $P = 1/5$. So

$$\frac{1/5}{4/5} = A\cdot 2^0 \quad \Rightarrow \quad A = \frac{1}{4}$$

$$\frac{P}{1-P} = \frac{1}{4}\cdot 2^t. \qquad \text{when } t=3,$$

$$\frac{P}{1-P} = \frac{1}{4}\cdot 2^3 = 2.$$

$$\frac{P}{1-P} = 2 \qquad P = 2 - 2P \qquad 3P = 2$$

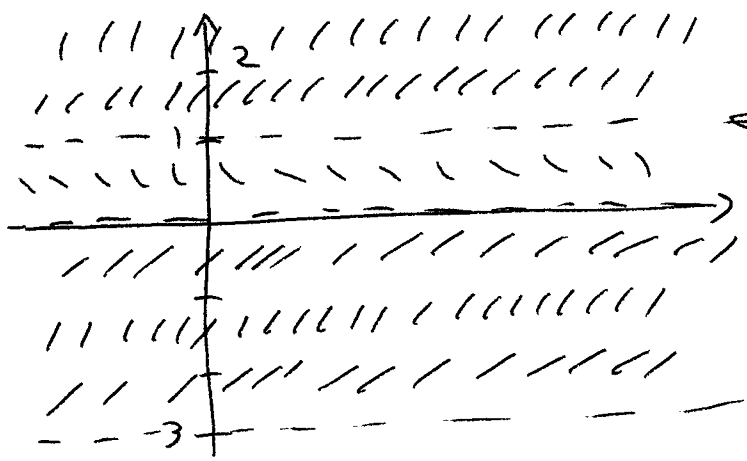
$$\boxed{P = \frac{2}{3}}$$

5. [20 points] Consider the differential equation $\frac{dy}{dx} = y^3 + 2y^2 - 3y$.

(a) Sketch the slope field of this equation, including equilibrium solutions.

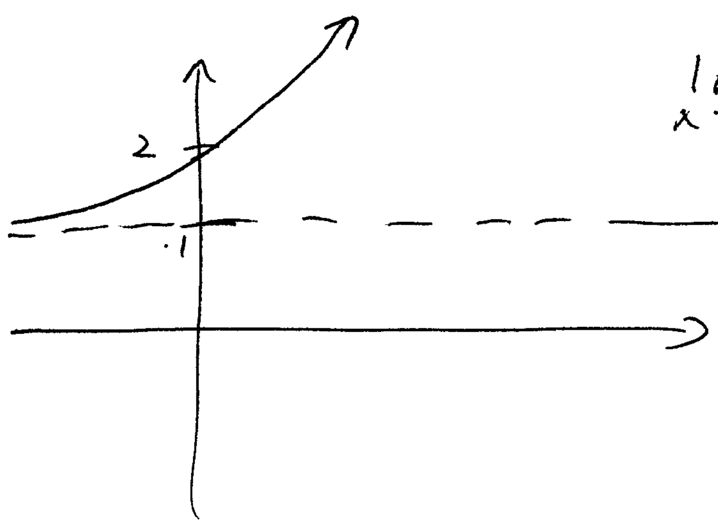
$\frac{dy}{dx} = y(y+3)(y-1)$.

y	$(-\infty,-3)$	$(-3,0)$	$(0,1)$	$(1,\infty)$
$\frac{dy}{dx}$	$-$	$+$	$-$	$+$



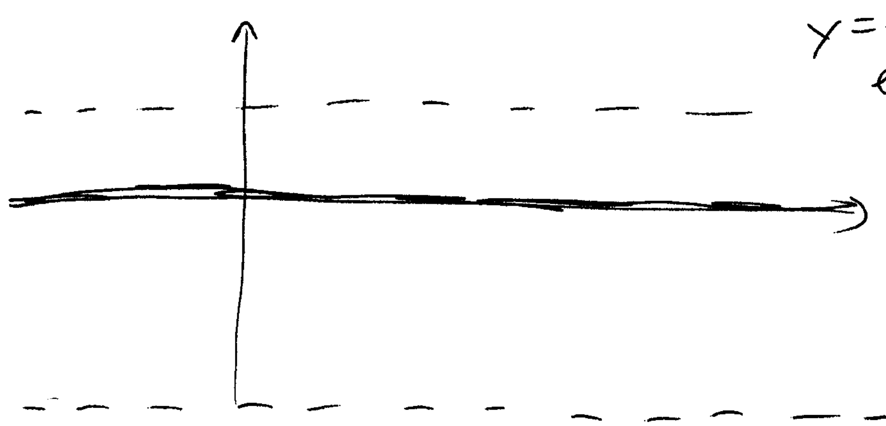
Equilibrium solutions: $y=0$, $y=1$, $y=-3$.

(b) Sketch the solution to the equation if $y(0) = 2$. What is $\lim_{x\to\infty} y(x)$?



$\lim_{x\to\infty} y(x) = \infty$

(c) Sketch the solution to the equation if $y(0) = 0$. What is $\lim_{x\to\infty} y(x)$?



$y=0$ is an equilibrium solution.

So $\lim_{x\to\infty} y(x) = 0$.