Midterm Exam 1

Math 153H, Spring 2008

You have 50 minutes. No notes, no books, no calculators. Good luck!

Name: Solutions ______________________

ID #: ______________________

1. __________ (/40 points)

2. __________ (/20 points)

3. __________ (/20 points)

4. __________ (/20 points)

Total __________ (/100 points)

1. [40 points] Evaluate the following integrals.

(a) $\int_0^1 (1-x)^9\,dx \quad = \quad -\int_1^0 u^9\,du \quad = \int_0^1 u^9\,du$

$u = 1-x$

$du = -dx$

$x = 0 \Rightarrow u = 1$

$x = 1 \Rightarrow u = 0$

$$= \left.\frac{u^{10}}{10}\right]_0^1$$

$$= \boxed{\frac{1}{10}}$$

(b) $\int x^2\,e^{-x}\,dx \longrightarrow \quad = -x^2e^{-x} + \int 2xe^{-x}\,dx$

$u = x^2 \qquad dv = e^{-x}dx$

$du = 2x\,dx \qquad v = -e^{-x}$

$u = 2x \qquad dv = e^{-x}dx$

$du = 2\,dx \qquad v = -e^{-x}$

$$= -x^2e^{-x} - 2xe^{-x} + \int 2e^{-x}\,dx$$

$$= -x^2e^{-x} - 2xe^{-x} - 2e^{-x} + C$$

$$= \boxed{e^{-x}(-x^2-2x-2) + C}$$

Question 1, continued.

(c) $\int \sin(x)\cos(\cos x)\,dx$ $= -\int \cos(u)\,du$

$u = \cos x$

$du = -\sin x\,dx$

$= -\sin(u) + C$

$= \boxed{-\sin(\cos x) + C}$

(d) $\int_1^4 \sqrt{x}\ln x\,dx$ $\longrightarrow$ $= \frac{2}{3}x^{3/2}\ln x\Big]_1^4 - \int_1^4 \frac{2}{3}x^{1/2}\,dx$

$u = \cancel{x}\ln x \qquad dv = x^{1/2}\,dx$

$du = \frac{1}{x}\,dx \qquad v = \frac{2}{3}x^{3/2}$

$= \left[\frac{2}{3}x^{3/2}\ln x - \frac{2}{3}\cdot\frac{2}{3}x^{3/2}\right]_1^4$

$= \frac{2}{3}\cdot 4^{3/2}\cdot\ln 4 - \frac{4}{9}\cdot 4^{3/2}$

$- \frac{2}{3}\cdot 1^{3/2}\cdot\ln(1) + \frac{4}{9}\cdot 1^{3/2}$

$= \frac{2}{3}\cdot 8\cdot 2\ln 2 - \frac{4}{9}\cdot 8$

$-0 + \frac{4}{9}$

$= \frac{32}{3}\ln 2 - \frac{32}{9} + \frac{4}{9}$

$= \boxed{\frac{32}{3}\ln 2 - \frac{28}{9}}$

Alternate approach:

$\int_1^4 x^{1/2}\ln x\,dx =$

$= \int_1^4 x^{1/2}\cdot\frac{2}{3}\ln\left(x^{3/2}\right)dx$

$= \int_1^8 \frac{4}{9}\ln(u)\,du,$

where $u = x^{3/2}$

$du = \frac{3}{2}x^{1/2}\,dx.$

2. [20 points] Let $f(\theta) = \dfrac{\sin\theta}{\cos^2\theta}$.

(a) Compute the average value of $f(\theta)$ on the interval $[0, \frac{\pi}{3}]$.

$$\overline{f(x)} = \frac{3}{\pi}\int_0^{\pi/3} \frac{\sin\theta}{\cos^2\theta}\,d\theta = \frac{-3}{\pi}\int_1^{\frac{1}{2}} \frac{1}{u^2}\,du$$

$u = \cos\theta$

$du = -\sin\theta\, d\theta$

$$= \frac{3}{\pi}\int_{1/2}^{1} \frac{1}{u^2}\,du$$

$$= \frac{-3}{\pi}\, u^{-1}\Big]_{1/2}^{1}$$

$$= \frac{-3}{\pi}\left(\frac{1}{1} - \frac{1}{1/2}\right)$$

$$= \frac{-3}{\pi}(-1)$$

$$= \boxed{3/\pi}$$

(b) Does the function f ever take this value on $[0, \frac{\pi}{3}]$? Explain.

Yes, by the <u>mean value theorem</u> for integrals.

Alternate answer: $f(0) = 0$, $f(\pi/3) = \dfrac{\sqrt{3}/2}{(1/2)^2}$

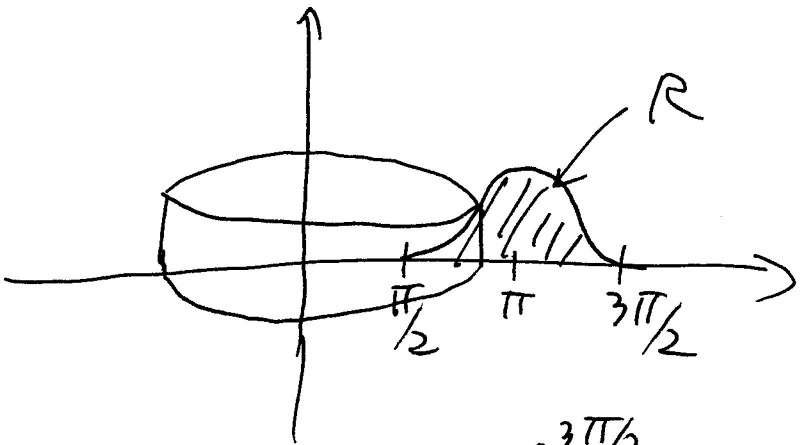
$= 2\sqrt{3}$.

So, since $0 < \overbrace{\frac{3}{\pi}}^{\approx 1} < \overbrace{2\sqrt{3}}^{>3}$,

$f(\theta)$ must equal $3/\pi$ for some θ by the <u>intermediate value theorem</u>.

3. [20 points] Let R be the region under the graph of $y = \cos^2 x$ on the interval $[\frac{\pi}{2}, \frac{3\pi}{2}]$. Let S be the solid obtained by revolving R around the y–axis.

(a) Set up an integral that expresses the volume of S. *You don't need to evaluate this integral.*



$$\mathrm{Vol}(S) = 2\pi \int_{\pi/2}^{3\pi/2} x \cos^2 x \, dx$$

(which can be done by parts)

(b) Use Pappus's theorem to compute the volume of S.

$$\begin{aligned}
\mathrm{Vol}(S) &= 2\pi \cdot \bar{x} \cdot \mathrm{Area}(R) \\
&= 2\pi \cdot \pi \cdot \int_{\pi/2}^{\frac{3\pi}{2}} \cos^2 x \, dx \\
&= 2\pi^2 \int_{\pi/2}^{\frac{3\pi}{2}} \frac{1+\cos(2x)}{2} \, dx \\
&= 2\pi^2 \left[\frac{x}{2} + \frac{\sin(2x)}{4} \right]_{\pi/2}^{3\pi/2} \\
&= 2\pi^2 \left(\frac{3\pi}{4} + \frac{\sin(3\pi)}{4} - \frac{\pi}{4} - \frac{\sin(\pi)}{4} \right) \\
&= 2\pi^2 \left(\frac{\pi}{2} \right) \\
&= \boxed{\pi^3}.
\end{aligned}$$

4. [20 points] Your cell phone has suddenly stopped allowing you to listen to your messages. So you have decided to call the wireless company to find out what's wrong. They tell you that the waiting times to speak to a person are distributed according to the function

$$f(t) = 3\,e^{-3t},$$

where t is measured in <u>hours</u>.

(a) What is the probability that you will wait less than 40 minutes?

$$P = \int_0^{2/3} 3e^{-3t}\,dt$$

$$= -e^{-3t}\Big]_0^{2/3}$$

$$= -e^{-2} + 1$$

$$= \boxed{1 - \frac{1}{e^2}} \approx 86.5\%$$

(b) Suppose that the company receives about $e^{10} \approx 22,000$ calls per day. About how many of these people will have to wait more than 3 hours?

$p(\text{wait more than 3 hours}) = 1 - p(\text{wait less than 3 hours})$

$$= 1 - \int_0^3 3e^{-3t}\,dt$$

$$= 1 + \left[e^{-3t}\right]_0^3$$

$$= 1 + e^{-9} - 1$$

$$= e^{-9}.$$

So the number of people waiting more than 3 hours is $\boxed{\frac{e^{10}}{e^9} = e \approx 3.}$