COST-MINIMIZING NETWORKS
AMONG IMMISCIBLE FLUIDS IN $\mathbb{R}^2$

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We model interfaces between immiscible fluids as cost-minimizing networks, where "cost" is a weighted length. We consider conjectured necessary and sufficient conditions for when a planar cone is minimizing. In some cases we give a proof; in other cases we provide a counterexample.

1. Introduction.

In planar networks of soap films, segments meet in triples to form 120° angles. Such angles are characteristic of nodes in length-minimizing networks. In this paper, we study the geometry of *cost*-minimizing planar networks, where we define cost to be weighted length. These networks have a much richer geometry than do soap-film networks. In particular, cost-minimizing networks can meet in any number around a node, with the angles between segments determined by their relative costs.

For examples of cost-minimizing networks, we look largely to immiscible fluids. When two immiscible fluids come together in a region of the plane, they meet to form an interface. This interface has an associated energy proportional to its length, which gives rise to a "cost constant." When several fluids come together, they form a network of interfaces, each interface with a specific cost constant. The fluids arrange themselves so as to minimize the total energy or cost. (Other generalizations of length-minimizing networks can be found in the surveys [**HRW**] and [**IT**].)

In general, one might impose area constraints on the regions occupied by the fluids. For this paper, though, we focus on the local behavior — how fluids come together at nodes. So we assume our networks are in the unit disk, and area constraints are replaced by constraints on how the regions meet the boundary circle. Our candidates for local minimizers are simply cones in the disk.

The conjecture.
We consider conjectured conditions for a cone to be minimizing. Conjecture 2.5 [**FMMP**, Conj. 2.7] states that a cone is minimizing if and only if a certain condition holds. This "calibration" condition involves placing a

point for each fluid, with constraints on the vectors between points. Gary
Lawlor and Frank Morgan proved that calibration is always sufficient for
minimization and necessary if all fluids are present (Theorem 2.2). They
leave open the question of whether the condition remains necessary if one
allows additional fluids. For simple cases, we prove that it does (Theorem
4.4). In general, however, we prove by counterexample that it does not
(Theorem 4.6). The primary tool that we use to prove the conjecture in
simple cases is Theorem 2.9, in which we consider an uncalibrated cone and
construct a cheaper competing network. To prove the counterexample, we
combine a modified calibration argument with an estimate on competing
networks obtained by filling regions with alternate fluids.

In the simplest case we consider, there is only one fluid absent from the
minimizing cone. In this case, we show that if the cone were not calibrated
we could insert a triangle of the extra fluid and decrease cost as in Figure 1.
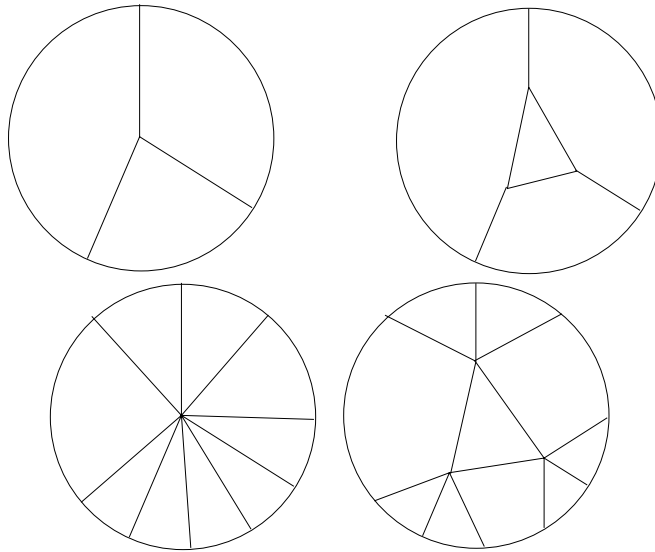

Figure 1. Inserting a triangular region of another fluid can
reduce cost.

The case of two fluids absent introduces new complexities, and includes
a counterexample. When the conjecture does hold, the competing network
required by Theorem 2.9 takes the form of two triangles sharing an edge as
in Figure 2. The counterexample arises when the two new fluids are only
helpful in reducing cost by bordering diagonally opposite original fluids.

When more than two fluids are absent, the problem becomes considerably
more complicated than the previous cases, even for just three fluids present
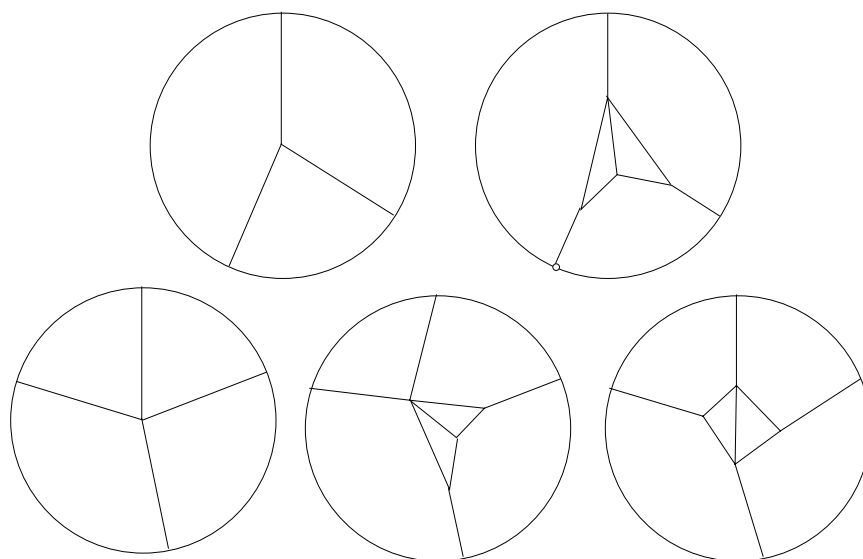and three absent [**FMMP**, §7]. In some subcases, it appears that we can

Figure 2. Inserting two triangular regions of other fluids can reduce cost.

prove that Conjecture 2.5 holds; in others, there will be counterexamples analogous to Theorem 4.6.

The counterexample.

A counterexample to Conjecture 2.5 must involve at least two absent and four present fluids (Theorem 4.4 and Remark 4.5). In fact, we use such a network; it is easy to show that the cone of our counterexample cannot be calibrated.

To show that our cone is minimizing, we need to show that adding the two absent fluids cannot reduce cost. We combine a modified calibration argument with calculations of the cost savings resulting from flooding competing networks with alternative fluids. The proof we give in this paper relies heavily on the flooding arguments to obtain contradictory estimates on interface lengths. An alternate proof involves a more elaborate calibration calculation, also taking into account the angles made by network boundaries [**GNY**, Thms. 1 and 2].

Our proof also helps to understand why a counterexample with two absent fluids requires at least four present fluids. The final contradiction we arrive at is that one extra fluid has to connect two diagonally opposite regions, and the other extra fluid has to connect the other diagonal pair. But this would require fluids to overlap, which is impossible.

Previous work.

The SMALL Geometry Groups 1995 [**BTW**] and 1996 [**EMN**] studied bubbles of immiscible fluids. Their work dealt with minimizing cost under area
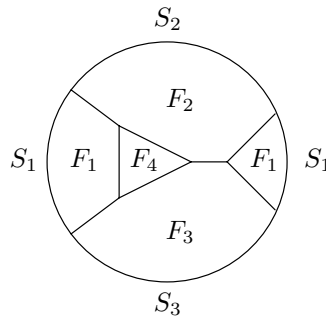
Figure 3. A network of six fluids, four of which are present $(R_5, R_6 = \emptyset)$.

constraints. By focusing on local behavior instead of entire bubbles, we avoid some of the difficulties they encountered. Brian Elieson [**E**] obtained sufficient conditions for a bound on the number of fluids meeting at a point.

2. Immiscible Fluids.

Section 2 begins with definitions, known existence and regularity (Theorem 2.1), and calibration. Our Theorem 2.9 uses calibration to prove that certain perturbations of a given network have less cost.

Definitions.
We consider disjoint, relatively open regions $R_1, \ldots, R_n$, associated with fluids $F_1, \ldots, F_n$, in the closed unit disk D. These regions are separated by a *network* N of C^1 curves meeting only at their endpoints at finitely many *nodes*, with $\bigcup R_i = D \setminus N$. (See Figure 3.)

Each region has a prescribed *boundary arc* $S_i = R_i \cap \partial D$, consisting of finitely many intervals. Fixed assignments of arcs S_i to fluids F_i are called *boundary conditions*. Each arc S_i may be empty, in which case R_i may or may not be empty. We will say that fluid F_i is *present* or *absent* according as R_i is nonempty or empty. In general, R_i may have several

components. For two regions R_i and R_j in a network N, we define the *interface* $N_{ij} = \partial R_i \cap \partial R_j$, oriented as the boundary of R_i. We may think of N as either the union or the collection of interfaces N_{ij}, as dictated by context.

To every pair of fluids F_i, F_j we assign a *cost constant* $a_{ij} = a_{ji} > 0$. We seek networks that minimize an energy or *cost*

$$(1) \qquad \text{cost}(N) \;\; = \;\; \sum_{i<j} a_{ij} \cdot \text{length}(N_{ij}).$$

We assume that the cost constants for any fluids F_i, F_j, F_k satisfy a *triangle inequality*

$$(2) \qquad\qquad a_{ik} \;\; \leq \;\; a_{ij} + a_{jk}.$$

(Otherwise, replacing N_{ik} by an infinitesimal layer of F_j would reduce cost.)

Existence and regularity of cost-minimizing networks.

Theorem 2.1 says that minimizing networks exist and consist of finitely many line segments.

Theorem 2.1 ([**M2**, Thm. 3.3 and Cor. 4.5])**.** *Let $F_1, \ldots, F_n$ be a set of fluids with cost constants a_{ij} that satisfy the strict triangle inequality ($a_{ik} < a_{ij} + a_{jk}$). For given boundary conditions, there exists a cost-minimizing network N. Furthermore, N consists of finitely many line segments.*

Frank Morgan [**M2**] and Brian White [**W1**], [**W2**] consider the more general problem with area constraints in the context of flat chains with co-efficients in a group representing the fluids. Morgan uses geometric measure theory to prove that the minimizer must consist of constant-curvature arcs. In our problem, where we do not place area constraints on the fluids, the constant-curvature arcs are of course straight lines. Brian Elieson [**E**] attempts a direct proof of the existence of straight-line minimizing networks, although he succeeds only assuming a bound on the number of nodes.

Calibration.

Consider a network N, consisting of linear interfaces N_{ij}. For each $N_{ij} \neq \emptyset$, construct a unit normal vector $n_{ij} \perp N_{ij}$, pointing from R_i into R_j. We say that points $p_1, \ldots, p_n$ *calibrate* N if

$$(3) \qquad\qquad p_j - p_i = a_{ij} n_{ij} \quad \text{when } N_{ij} \neq \emptyset, \text{ and}$$

$$(4) \qquad\qquad |p_j - p_i| \;\leq\; a_{ij} \quad \text{when } N_{ij} = \emptyset.$$

From now on we focus our attention on the local behavior of a network in the neighborhood of a node, i.e., a *cone*, consisting of line segments meeting at the origin. Gary Lawlor and Frank Morgan [**LM**] prove that such networks minimize cost if and only if they are calibrated, assuming that all fluids are present.

Theorem 2.2 ([**LM**, Thms. 2.1 and 2.5])**.** *Let N be a cone. If N is calibrated, it is cost minimizing. Conversely, if N is cost minimizing and each fluid is present, then N can be calibrated.*

Remark 2.3. If each fluid is present, then by (3) the points p_i are unique, up to an additive constant.

Lemma 2.4. *Consider a cone N calibrated by points $p_1, \ldots, p_n$. Then the polygon $p_1 \cdots p_n$ is strictly convex and each R_i is connected.*

Proof. Since N is calibrated and hence minimizing, it follows easily that the angle between consecutive lines of N is less than $180°$. Since $p_j - p_i$ is normal to N_{ij}, it follows that $p_1 \cdots p_n$ is strictly convex and that each R_i is connected. $\square$

The conjecture we explore in this paper seeks to extend Theorem 2.2 and show that minimizing always implies calibrated.

Conjecture 2.5. *A minimizing cone can be calibrated.*

Remark 2.6. For one fluid present, the conjecture is trivially true. For two fluids present, the triangle inequality implies the conjecture [**E**, Prop. 6.1]. Thus, we will only consider cases with at least three fluids present.

Remark 2.7. Conjecture 2.5 holds for length-minimizing networks, when all cost constants $a_{ij} = 1$. It is well known that for such networks, segments meet in triples to form $120°$ angles ([**HRW**, §1.3, Thm. 1.1, p. 6], [**IT**, Ch. 3, Thm. 2.1, p. 120]). Such cones have an essentially unique calibration consisting of an equilateral triangle, and points for the absent fluids can be placed anywhere inside this triangle.

We will later show that this conjecture is false (Theorem 4.6), which leads us to the following open question.

Question 2.8. When can a minimizing cone be calibrated?

Competing networks.

The following theorem shows how to reduce the cost of certain networks by inserting absent fluids at a node. It will be used to prove in Theorems 3.1 and 4.4 that certain uncalibrated cones are not minimizing.

Theorem 2.9. *Consider a smooth family $\{N(t)\}$ of networks of line segments $N_{ij}(t)$, $0 \le t \le T$ all satisfying the same boundary conditions, such that unless $i, j \le m < n$, $N_{ij}(0)$ is empty and for all $0 < t_1, t_2 \le T$, $N_{ij}(t_2)$ is parallel to $N_{ij}(t_1)$. Suppose $N(0)$ is calibrated by $p_1, \ldots, p_m$ for fluids $F_1, \ldots, F_m$, and suppose there are points $p_{m+1}, \ldots, p_n$ such that*

- *when $i, j \le m$ and $N_{ij}(t) \ne \emptyset$, then $(p_j - p_i) \cdot n_{ij}(t) = a_{ij} \cos \theta_{ij}(t)$ (where $\theta_{ij}(t)$ is typically some small angle and $\theta_{ij}(0) = 0$ because $N(0)$ is calibrated by $p_1, \ldots, p_n$); and*

- when $i > m$ or $j > m$ and $N_{ij}(t) \neq \emptyset$, then $p_j - p_i = \alpha_{ij} n_{ij}$, where $\alpha_{ij} \geq a_{ij}$ and $\alpha_{bc} > a_{bc}$ for some $N_{bc}(t)$ of length t. (*The normal n_{ij}, and therefore α_{ij} as well, is independent of t, since $N_{ij}(t_2)$ is parallel to $N_{ij}(t_1)$.*)

Then, for small positive t, $\mathrm{cost}(N(t)) < \mathrm{cost}(N(0))$. In particular, $N(0)$ is not a minimizing network.

Proof. Let $|N_{ij}(t)|$ denote the length of $N_{ij}(t)$. Then

$$
\begin{aligned}
\mathrm{cost}(N(0)) &= \sum_{i<j\leq m} a_{ij}\, |N_{ij}(0)| \\
&= \sum_{i<j\leq m} \mathrm{flux}_{N_{ij}(0)}(p_j - p_i) \\
&= \sum_{i=1}^{m} \mathrm{flux}_{S_i}(p_i) \\
&= \sum_{i=1}^{n} \mathrm{flux}_{S_i}(p_i) \\
&= \sum_{i<j\leq n} \mathrm{flux}_{N_{ij}(t)}(p_j - p_i),
\end{aligned}
$$

and

$$
\begin{aligned}
\mathrm{cost}(N(t)) &= \sum_{i<j\leq n} a_{ij}|N_{ij}(t)| \\
&= \sum_{i<j\leq m} a_{ij}|N_{ij}(t)| + \sum_{\substack{i<j \\ m<j}} a_{ij}|N_{ij}(t)|.
\end{aligned}
$$

Combining these two equations, we get

$$
\begin{aligned}
\mathrm{cost}(N(t)) - \mathrm{cost}(N(0)) &= \sum_{i<j\leq m} \left(a_{ij}|N_{ij}(t)| - \mathrm{flux}_{N_{ij}(t)}(p_j - p_i)\right) \\
&\quad + \sum_{\substack{i<j \\ m<j}} \left(a_{ij}|N_{ij}(t)| - \mathrm{flux}_{N_{ij}(t)}(p_j - p_i)\right) \\
&= \sum_{i<j\leq m} a_{ij}(1 - \cos\theta_{ij})|N_{ij}(t)| \\
&\quad + \sum_{\substack{i<j \\ m<j}} (a_{ij} - \alpha_{ij})|N_{ij}(t)|.
\end{aligned}
$$

Now we can calculate the derivative of $\mathrm{cost}(N(t))$ at $t = 0$. Here, we must recall that $\theta_{ij}(0) = 0$. Recall also that $|N_{ij}(0)| = 0$ if $j > m$. Since

$|N_{ij}|'(0) \geq 0$ for all $j > m$, therefore every $(a_{ij} - \alpha_{ij})|N_{ij}|'(0) \leq 0$. Furthermore, since some $\alpha_{bc} > a_{bc}$, this $(a_{bc} - \alpha_{bc})|N_{bc}|'(0) < 0$. Hence

$$
\begin{aligned}
\frac{d}{dt}\bigg|_{t=0} &\ \mathrm{cost}(N(t)) \\
&= \frac{d}{dt}\bigg|_{t=0} \mathrm{cost}(N(t)) - \mathrm{cost}(N(0)) \\
&= \sum_{i<j\leq m} a_{ij}\Big(\sin\theta_{ij}(0)\,\theta'_{ij}(0)\,|N_{ij}(0)| \\
&\qquad + (1 - \cos\theta_{ij}(0))\,|N_{ij}|'(0) \Big) + \sum_{\substack{i<j \\ m<j}} (a_{ij} - \alpha_{ij})|N_{ij}|'(0) \\
&= \sum_{\substack{i<j \\ m<j}} (a_{ij} - \alpha_{ij})|N_{ij}|'(0) \\
&< 0.
\end{aligned}
$$

Therefore, for sufficiently small $t > 0$, $N(t)$ is a cheaper network than $N(0)$. $\qquad\square$

3. One fluid absent.

Theorem 3.1 considers a minimizing network N with one fluid absent — say F_n — and applies Theorem 2.9 to prove that N can be calibrated. By Theorem 2.2, we already know there is a calibration $p_1, \dots, p_{n-1}$ for fluids $F_1, \dots, F_{n-1}$. To place p_n within a_{in} of p_i for $i \leq n-1$, we just need to know that the intersection of the disks $D_i = \mathrm{D}(p_i, a_{in})$ is nonempty (see Figure 4).

Theorem 3.1. *Let N be a minimizing cone. If at most one fluid is absent, then N can be calibrated.*

Proof. If no R_i is empty, the result reduces to Theorem 2.2. Thus we may suppose without loss of generality that R_n is empty. We will show that if N cannot be calibrated, then it must not be a minimizer, by constructing a cheaper network with a nonempty R_n.

Suppose N cannot be calibrated. By definition, the intersection of the disks $D_i = \mathrm{D}(p_i, a_{in})$ is empty. Then, by Helly's Theorem [**Lay**, Thm. 6.2], there exist $1 < b < c \leq n-1$ (possibly after relabeling such that $D_1 \cap D_b \cap D_c = \emptyset$. Place p_n in the region omitted by the three disks; note that $p_n \in \triangle p_1 p_b p_c$.

For small $t > 0$, construct a triangle

$$
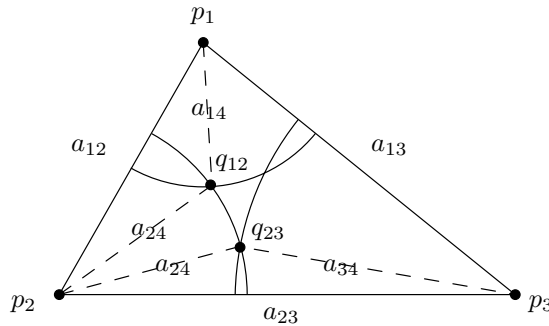T(t) \quad = \quad \{N_{1n}(t), N_{bn}(t), N_{cn}(t)\},
$$

Figure 4. The disks $D(p_i, a_{i4})$ must intersect pairwise because of the triangle inequality. However if all three have no common intersection, point p_4 cannot be placed close enough to the others to satisfy (4). So the corresponding three-fluid network will be minimizing if and only if this common intersection of the three disks $D(p_i, a_{i4})$ is nonempty; this happens if and only if $\angle p_3 p_2 q_{23} + \angle p_1 p_2 q_{12} \geq \angle p_1 p_2 p_3$.

where these $N_{in}(t)$ are normal to the segments $\overline{p_i p_n}$ as in Figure 5 and $t = |N_{1n}(t)|$. Since $p_n \in \triangle p_1 p_b p_c$, $\partial T(t)$ consists of, in counterclockwise order, $N_{1n}(t)$, $N_{bn}(t)$, and $N_{cn}(t)$. Translate $T(t)$ so that its center of mass lies at the origin; then $T(t)$ shrinks to the origin as $t \to 0$.

Now add segments $N_{i(i+1)}(t)$ connecting $N_{i(i+1)} \cap \partial D$ (the prescribed boundary) to the appropriate vertices of $T(t)$. For $1 \leq i < b$, this will be $N_{1n}(t) \cap N_{bn}(t)$; for $b \leq i < c$, this will be $N_{bn}(t) \cap N_{cn}(t)$; and for $c \leq i \leq n - 1$, this will be $N_{cn}(t) \cap N_{1n}(t)$, as in Figure 5.

Then we have a family of networks $N(t) = T(t) \cup \{N_{i(i+1)}(t)\}$, with fluid F_n filling the interior of $T(t)$. It follows by Theorem 2.9 that $N = N(0)$ is not minimizing. $\square$

An algebraic criterion for one of four fluids absent.
Theorems 2.2 and 3.1 (with $n = 3$) establish necessary and sufficient conditions for whether a calibrated cone N separating three fluids can be improved by adding a fourth fluid. Theorem 3.2 reduces those conditions to an algebraic inequality on the cost constants a_{ij}.

Theorem 3.2. *Consider a minimizing network N consisting of three line segments meeting at a node, separating fluids F_1, F_2, F_3. Its cost can be reduced by adding a fourth fluid if and only if*

$$(5) \quad \cos^{-1}\left(\frac{a_{23}^2 + a_{24}^2 - a_{34}^2}{2a_{23}a_{24}}\right) + \cos^{-1}\left(\frac{a_{12}^2 + a_{24}^2 - a_{14}^2}{2a_{12}a_{24}}\right)$$
$$< \cos^{-1}\left(\frac{a_{12}^2 + a_{23}^2 - a_{13}^2}{2a_{12}a_{23}}\right).$$
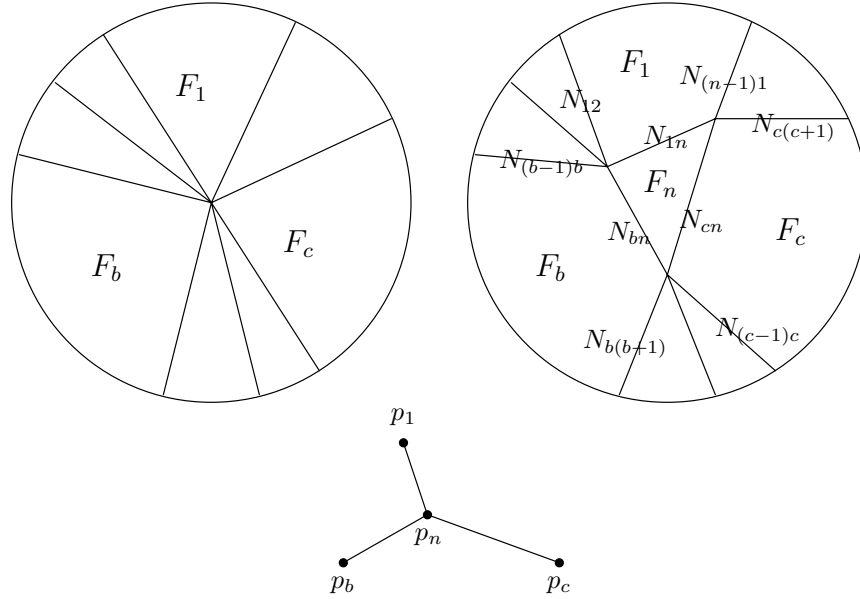
Figure 5. Starting with an uncalibrated network (left), we know that $\triangle p_1 p_b p_c$ contains p_n (bottom). Hence we can construct a comparison network (right) containing the triangle $T(t) = \{N_{1n}(t), N_{bn}(t), N_{cn}(t)\}$. ($N_{ij}(t)$ separates fluids F_i and F_j; for simplicity we have dropped the "(t)"s in the diagram.) This comparison network beats the original network.

Proof. In light of Theorems 2.2 and 3.1, it suffices to show that inequality (5) holds if and only if the essentially unique calibration p_1, p_2, p_3 for fluids F_1, F_2, F_3 cannot be extended to a calibration $p_1, \ldots, p_4$ for fluids $F_1, \ldots, F_4$, i.e., if and only if the intersection E of the three disks $\mathrm{D}(p_i, a_{i4})$ is empty.

Letting $q_{ij} = \partial\mathrm{D}(p_i, a_{i4}) \cap \partial\mathrm{D}(p_j, a_{j4})$ (see Figure 4), we see that this occurs if and only if

$$\angle p_3 p_2 q_{23} + \angle p_1 p_2 q_{12} \; < \; \angle p_1 p_2 p_3.$$

By the law of cosines, this reduces to (5). $\qquad\square$

Remark 3.3. In the symmetric case where $a_{12} = a_{13} = a_{23} = 1$ and $a_{14} = a_{24} = a_{34} = a$, the inequality (5) reduces to

$$(6) \qquad\qquad 2\cos^{-1}\left(\frac{1}{2a}\right) < \cos^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{3}.$$

This holds only when $\cos^{-1}\left(\frac{1}{2a}\right) < \frac{\pi}{6}$, or $a < \frac{1}{\sqrt{3}}$. Students at the SUNY Geneseo Student Workshop in April 1997 [**M1**] proved this case.

The Geneseo students also considered a symmetric case of adding a fifth fluid to a minimizing cone separating F_1, F_2, F_3, F_4. Here, $a_{12} = a_{23} = a_{34} = a_{34} = 1$, $a_{13} = a_{24} = \sqrt{2}$, and $a_{i5} = a$. In this case, the original network can be made cheaper by adding the fifth fluid if $a < \frac{1}{\sqrt{2}}$, and (3) and (4) can be satisfied if $a \geq \frac{1}{\sqrt{2}}$. Extending the results of this section to that case would be complicated.

4. Two fluids absent.

Section 4 considers a minimizing cone N of n fluids, with two fluids absent, say F_{n-1} and F_n. Theorem 4.4 proves that N can be calibrated, with one possible exception. By Remark 2.6 we can assume $n \geq 5$; the exception first occurs when $n = 6$ (four fluids present, two absent). Theorem 4.6 will later show that the exceptional case yields a counterexample. Before showing that minimizing implies calibrated (with the stated exception), we must first introduce a few concepts.

"Eyes".

An *eye* is the intersection of (closed) disks. An *arc* of the eye is a circular arc making up part of the boundary of the eye. A *vertex* of the eye is an intersection of two arcs.

Define the two eyes

$$E_{n-1} = \bigcap_{i=1}^{n-2} D(p_i, a_{i(n-1)}) \qquad \text{and} \qquad E_n = \bigcap_{i=1}^{n-2} D(p_i, a_{in}).$$

We will call p_i a *cost-realizing point* for $p_n \in E_n$ if $|p_i - p_n| = a_{in}$.

We wish to show that (with the stated exception) any minimizing network can be calibrated. To do this, we will show that any network that cannot be calibrated can be beaten. By Theorem 2.2, any network without a calibration for $F_1, \ldots, F_{n-2}$ can be beaten (using only fluids $F_1, \ldots, F_{n-2}$). By Theorem 3.1, any network with $E_{n-1} = \emptyset$ or $E_n = \emptyset$ can be beaten (using only $n - 1$ fluids). Therefore, for the purposes of the following lemmas, we assume that

- $p_1, \ldots, p_{n-2}$ calibrate $F_1, \ldots, F_{n-2}$; and
- E_{n-1} and E_n are nonempty.

Let us place $p_{n-1} \in E_{n-1}$ and $p_n \in E_n$ so as to minimize $|p_{n-1} - p_n|$. If there is no calibration of N for all n fluids $F_1, \ldots, F_n$, then

(7) $$|p_{n-1} - p_n| > a_{(n-1)n}.$$

Lemma 4.1. *The points $p_{n-1} \in E_{n-1}$ and $p_n \in E_n$ which minimize $|p_{n-1} - p_n|$ lie on vertices of E_{n-1} and E_n.*

Proof. Let $\alpha = |p_{n-1} - p_n|$. Clearly $D(p_{n-1}, \alpha)$ and E_n meet only at a single point, which is p_n. If p_n is on the interior of the p_i-arc of the eye E_n, then

the two circles $\mathrm{circ}(p_{n-1}, \alpha)$ and $\mathrm{circ}(p_i, a_{in})$ are tangent. It follows that $p_n \in \overline{p_i p_{n-1}}$. By inequality (7), $\alpha > a_{(n-1)n}$ and

$$
\begin{aligned}
a_{i(n-1)} & \geq |p_i - p_{n-1}| \\
& = |p_i - p_n| + |p_n - p_{n-1}| \\
& = a_{in} + \alpha \\
& > a_{in} + a_{(n-1)n},
\end{aligned}
$$

contradicting the triangle inequality for cost constants. So p_n cannot be in the interior of an arc on the eye, and it must be at a vertex. Similarly, p_{n-1} must be on a vertex of E_{n-1}. $\qquad\square$

Lemma 4.2. *Suppose that $p_{n-1} \in E_{n-1}$ and $p_n \in E_n$ are placed as close together as possible. Then there exist two cost-realizing points p_i and p_j for p_{n-1} such that $p_{n-1} \in \triangle p_i p_j p_n$.*

Proof. Consider the convex hull of p_n along with all of the cost-realizing points for p_{n-1}. (By Lemma 4.1 there are at least two cost-realizing points for p_{n-1}.) If p_{n-1} were not inside this convex hull, moving it closer to the nearest point of the convex hull would decrease the distance between p_{n-1} and all of the points involved in the convex hull (i.e., p_n and the cost-realizing points). Since we chose $p_{n-1} \in E_{n-1}$ to be as close to p_n as possible, this cannot happen. Hence p_{n-1} lies inside the convex hull of p_n and the cost-realizing points. Since this is just the union of triangles formed by p_n and pairs of cost-realizing points, there is a pair $\{p_i, p_j\}$ of cost-realizing points such that $p_{n-1} \in \triangle p_n p_i p_j$. $\qquad\square$

We will say that p_{n-1} lies on the p_i-p_j vertex of its eye E_{n-1} if $p_{n-1} \in E_{n-1}$ and $p_n \in E_n$ are placed as closed together as possible and p_i and p_j satisfy the conclusion of Lemma 4.2, i.e., if p_i and p_j are cost-realizing points for p_{n-1} such that $p_{n-1} \in \triangle p_i p_j p_n$.

Lemma 4.3. *Suppose $p_{n-1} \in E_{n-1}$ and $p_n \in E_n$ have been chosen to minimize $|p_{n-1} - p_n|$. Then p_{n-1} and p_n cannot both be on p_b-p_c vertices of their eyes.*

Proof. If both p_{n-1} and p_n are on p_b-p_c vertices of E_{n-1} and E_n, then by Lemma 4.2, $p_{n-1} \in \triangle p_b p_c p_n$ and $p_n \in \triangle p_b p_c p_{n-1}$. But this can hold only if p_b, p_c, p_{n-1}, p_n are collinear, which cannot happen by triangle inequality considerations similar to those in Lemma 4.1. It follows that p_{n-1} and p_n cannot both be on p_b-p_c vertices. $\qquad\square$

Two fluids absent.

We can now state and prove our theorem about two fluids absent. It is here that we encounter a counterexample to Conjecture 2.5. We do, however, prove that many types of minimizing networks can be calibrated. Theorem 4.6 provides a counterexample to Conjecture 2.5 at the end of the section.

Suppose that $p_{n-1} \in E_{n-1}$ and $p_n \in E_n$ are placed as close together as possible, with p_{n-1} on the p_b-p_c vertex of E_{n-1} and p_n on the p_d-p_e vertex of E_n. Let us renumber $p_1, \ldots, p_{n-2}$ in a counterclockwise order, so that (without loss of generality) $b < c$ and $d < e$. We will say that p_{n-1} and p_n are on *opposite vertices* of their eyes if $b < d < c < e$.

Theorem 4.4. *Consider a minimizing cone N. If at most two regions R_i are empty, then N can be calibrated, with the following possible exception: placing the points — say, p_{n-1} and p_n — for two absent fluids as close as possible while still close enough to the essentially uniquely placed other p_i puts them on opposite vertices of the eyes E_{n-1} and E_n.*

Remark 4.5. When $n = 5$, there are no opposite vertices for p_{n-1} and p_n to lie on since all fluids in the original network are adjacent. However, when $n \geq 6$, this situation can occur, and Theorem 4.6 provides a counterexample to Conjecture 2.5.

Proof. If at most one R_i is empty, the result reduces to Theorem 3.1. Thus we may suppose without loss of generality that regions R_{n-1} and R_n are empty. We proceed by contradiction, supposing that N cannot be calibrated and that the exceptional condition does not hold.

Recall that $p_1, \ldots, p_{n-2}$ are the essentially unique calibration for fluids $F_1, \ldots, F_{n-2}$ and that $p_{n-1} \in E_{n-1}$ and $p_n \in E_n$ have been chosen to minimize the distance between them. To derive a contradiction, we will assume that inequality (7) holds and then produce a family $\{N(t)\}$ of competing networks of less cost.

By Lemma 4.2, p_{n-1} lies on a p_b-p_c vertex and p_n lies on a p_d-p_e vertex. Relabel if necessary so that $p_1, \ldots, p_{n-2}$ are in counterclockwise order (so the cone N consists of interfaces $N_{i(i+1)}$ with indices taken modulo $n-2$) with $1 = b < c$ and $d < e$. By Lemma 4.3, we may take $b < d$ as well. We do not consider $b < d < c < e$, so we have only the cases where $b < c \leq d < e$.

Now, we know that the polygon $p_1 \cdots p_{n-2}$ is convex by Lemma 2.4. Also, $p_{n-1} \in \triangle p_1 p_c p_n$ and $p_n \in \triangle p_d p_e p_{n-1}$ by Lemma 4.2. Hence the four polygons at the top of Figure 6,

$$
\begin{aligned}
P_1 &= p_1 p_2 \cdots p_{c-1} p_c p_{n-1}, \\
P_2 &= p_c p_{c+1} \cdots p_{d-1} p_d p_n p_{n-1}, \\
P_3 &= p_d p_{d+1} \cdots p_{e-1} p_e p_n, \text{ and} \\
P_4 &= p_e p_{e+1} \cdots p_{n-2} p_1 p_{n-1} p_n,
\end{aligned}
$$

are all convex, regardless of whether $c = d$ (in which case P_2 is just a triangle) or $c < d$.

For $t > 0$ small, construct a split quadrilateral

$$Q(t) = \{N_{1(n-1)}(t), N_{c(n-1)}(t), N_{(n-1)n}(t)\} \cup \{N_{(n-1)n}(t), N_{dn}(t), N_{en}(t)\}$$
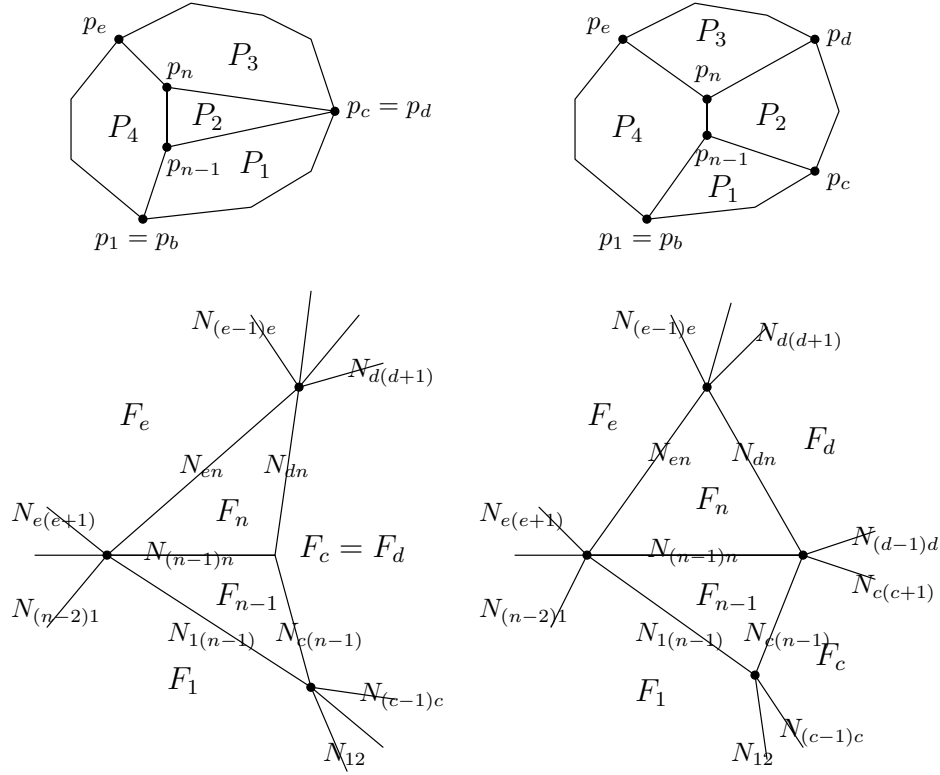
Figure 6. Since the polygons P_1, P_2, P_3, P_4 (top) are convex, we can construct a comparison network containing the quadrilateral $Q(t)$ (bottom). ($N_{ij}(t)$ separates fluids F_i and F_j; for simplicity we have dropped the "(t)"s in the diagram.) This works regardless of whether $c = d$ (left) or $c < d$ (right).

composed of two triangles sharing an edge, with these $N_{ij}(t)$ normal to the segments $\overline{p_i p_j}$ (see Figure 6, bottom) and $t = |N_{(n-1)n}(t)|$. The convexity of polygons P_i ensures that the two triangles

$$\triangle_{n-1}(t) = \{N_{1(n-1)}(t), N_{c(n-1)}(t), N_{(n-1)n}(t)\}$$
$$\triangle_n(t) = \{N_{(n-1)n}(t), N_{dn}(t), N_{en}(t)\}$$

are disjoint except for their shared edge $N_{(n-1)n}(t)$; and also ensures that $\partial Q(t)$ consists of, in counterclockwise order, the edges $N_{1(n-1)}(t)$, $N_{c(n-1)}(t)$, $N_{dn}(t)$, $N_{en}(t)$. Furthermore, translate $Q(t)$ so that its center of mass lies at the origin; then, as $t \to 0$, $Q(t)$ shrinks to the origin.

Now, add segments $N_{i(i+1)}(t)$ connecting $N_{i(i+1)} \cap \partial D$ (the prescribed boundary) to the appropriate vertices of $Q(t)$. For $1 \leq i < c$, this will
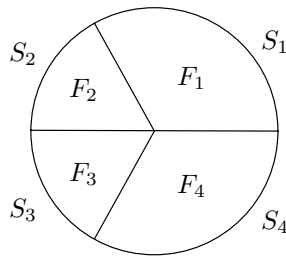
Figure 7. A counterexample to Conjecture 2.5, consisting of four present fluids and two absent. The network is minimizing but cannot be calibrated.

be $N_{1(n-1)}(t) \cap N_{c(n-1)}(t)$; for $c \leq i < d$ (if such i exist), this will be $N_{c(n-1)}(t) \cap N_{dn}(t)$; for $d \leq i < e$, this will be $N_{dn}(t) \cap N_{en}(t)$; and for $e \leq i \leq n-2$, this will be $N_{en}(t) \cap N_{1(n-1)}(t)$. (See Figure 6, bottom.)

Then we have a family of networks $N(t) = Q(t) \cup \{N_{i(i+1)}(t)\}$, with fluid F_{n-1} filling the interior of $\triangle_{n-1}(t)$ and fluid F_n in the interior of $\triangle_n(t)$. It follows by Theorem 2.9 that $N = N(0)$ is not minimizing. $\square$

A counterexample to the conjecture.

Finally, we prove a counterexample to Conjecture 2.5.

Theorem 4.6. *Consider the network N consisting of four straight lines meeting at a node separating a disc into four regions of fluids F_1, F_2, F_3, F_4 as in Figure 7. There exist cost constants a_{ij} $(1 \leq i, j \leq 6)$ satisfying the strict triangle inequality such that N cannot be calibrated, but is still minimizing.*

Proof. We define six points $p_1, p_2, \ldots, p_6$ in $\mathbb{R}^2$ as in Figure 8:

$$
\begin{aligned}
p_1 &= (\cos 30°, 2) \\
p_2 &= (0, 1.5) \\
p_3 &= (0, 0.5) \\
p_4 &= (\cos 30°, 0) \\
p_5 &= (\cos 30° - 0.11, 0.94) \\
p_6 &= (\cos 30° - 0.11, 1.06).
\end{aligned}
$$

The cost constants are given by $a_{ij} = |p_i - p_j| + g(i,j)\epsilon$, where ϵ is a small positive constant chosen to be 10^{-9} and $g(i,j)$ is defined as follows:

$$
g(i,j) = \begin{cases} 60000 & \text{if } (i,j) = (1,5), (3,5), (2,6) \text{ or } (4,6) \\ -1 & \text{if } (i,j) = (5,6) \\ 0 & \text{otherwise.} \end{cases}
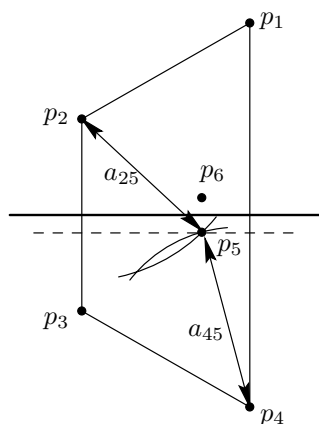$$

Figure 8. The points p_1, p_2, p_3, p_4 calibrate the network for four fluids, but an attempted placement of a point q_5 for fluid F_5 must lie below the dotted line. The heavy line is the axis of symmetry.

The numerical values of the cost constants are given below.

$$a_{12} = a_{23} = a_{34} = 1$$
$$a_{13} = a_{24} = \sqrt{3} \approx 1.73205080757$$
$$a_{14} = 2$$
$$a_{15} = a_{46} \approx 1.06575226327$$
$$a_{16} = a_{45} \approx 0.94641428561$$
$$a_{25} = a_{36} \approx 0.94083708003$$
$$a_{26} = a_{35} \approx 0.87480248277$$
$$a_{56} = 0.119999999.$$

It is easy to check by hand that strict triangle inequalities are satisfied. Alternatively, observe that the cost constants are equal to the Euclidean distances between corresponding points, sometimes with small corrections. The triangle inequality for the Euclidean metric thus implies strict triangle inequality for our cost constants with ϵ small, since no three points p_i, p_j, p_k lie on the same line.

Now, we claim that no calibration for $F_1, F_2, \ldots, F_6$ exists for network N. For suppose that $\{q_1, q_2, \ldots, q_6\}$ calibrated N with respect to $F_1, F_2, \ldots, F_6$. Since $a_{ij} = |p_i - p_j|$ for $1 \leq i < j \leq 4$, the placing of points q_1, q_2, q_3 and q_4 is essentially unique. So we may assume q_1, q_2, q_3, q_4 coincide with p_1, p_2, p_3, p_4, respectively, as in Figure 8. Since $a_{25} = |p_2 - p_5|$ and $a_{45} = |p_4 - p_5|$, we conclude that q_5 lies (horizontally) below p_5. Similarly, q_6 must lie above p_6.

Since p_5 and p_6 lie on the same vertical line, $|q_5 - q_6| \geq |p_5 - p_6| = a_{56} + \epsilon > a_{56}$, and it follows that N cannot be calibrated.

The difficult part is to show that N is cost-minimizing for $F_1, F_2, \ldots, F_6$. Suppose not; let M be a minimizing network. Since by Theorem 2.2 N is minimizing for fluids $\{F_1, F_2, F_3, F_4, F_5\}$ and for $\{F_1, F_2, F_3, F_4, F_6\}$, both F_5 and F_6 must be present in the network M.

In the network M with interfaces M_{ij}, let R_i denote the region of fluid F_i.

The main technique of our proof will be a process called *flooding*. We consider the alternative network formed when we flood a particular region of M with a different fluid. Since M is already cost-minimizing, this process cannot reduce cost. We will use this fact to generate bounds on the lengths of various interfaces.

The next step of our proof is to show that the region R_1 of fluid F_1 is connected. Of course, the boundary conditions force R_1 to be nonempty. But if there were a second component of R_1 disjoint from S_1, then flooding it with F_6 would reduce cost, since $a_{j6} < a_{1j}$. It follows that there is only one connected region of fluid F_1. Similarly, we see that R_2, R_3 and R_4 are all connected (by flooding with F_6, F_5 and F_5, respectively).

Now the following Lemma 4.7 yields a contradiction, as desired. $\square$

Lemma 4.7. *In the minimizing network M (with both F_5 and F_6 present),*

- *Some component of R_5 is adjacent to both R_2 and R_4.*
- *Some component of R_6 is adjacent to both R_1 and R_3.*

Proof. To begin with, we need a few definitions. For each region R_i, let $|M_i| = |M_{i5}| + |M_{i6}|$ denote the total length of the boundary that R_i shares with the two additional fluids F_5 and F_6. Define the three numerical constants:

$$C_1 \;=\; g(1,5) = g(3,5) = g(2,6) = g(4,6) \;=\; 60000,$$

$$C_2 \;=\; 1 + \frac{a_{14} + a_{23} - 2a_{45}}{2a_{35} - a_{13}} \;\approx\; 64.07174803043, \text{ and}$$

$$C_3 \;=\; \max\{a_{16} - a_{15}, a_{26} - a_{25}, a_{36} - a_{35}, a_{46} - a_{45}\} \;\approx\; 0.11933797766.$$

Now, suppose that no component of R_5 is adjacent to both R_2 and R_4. (The other case can be dealt with symmetrically.) First we use flooding to show that $|M_{56}|$ is relatively small.

By flooding R_5 with fluid F_6, we find that

$$(8) \qquad |M_{56}| \;\leq\; C_3 \frac{|M_1| + |M_2| + |M_3| + |M_4|}{a_{56}}.$$

By flooding $R_5 \cup R_6$ with F_1 and then with F_3, we calculate that

$$(9) \qquad |M_1| + |M_2| + |M_3| + |M_4| \;\leq\; C_2\,(|M_2| + |M_4|).$$

Let U be a connected component of R_5 that is adjacent to R_2. By assumption U is not adjacent to R_4, so by flooding U with F_6 and using $|U_1| + |U_3| + |U_6| \geq |U_2|$ we find that

$$|U_1| + |U_3| \geq \frac{a_{56} + a_{25} - a_{26}}{a_{56} + a_{36} - a_{35}} |U_2| \geq \frac{a_{56} - C_3}{a_{56} + C_3} |U_2|,$$

where $|U_i|$ denotes the length of the interface $\partial U \cap \partial R_i$. Of course, this inequality still holds (trivially) for a connected component U of R_5 that is not adjacent to R_2, since then $|U_2| = 0$. So by summing over all components of R_5, we get

$$(10) \qquad |M_{25}| \leq \frac{a_{56} + C_3}{a_{56} - C_3} \left(|M_{15}| + |M_{35}| \right).$$

Similarly, we obtain

$$(11) \qquad |M_{45}| \leq \frac{a_{56} + C_3}{a_{56} - C_3} \left(|M_{15}| + |M_{35}| \right).$$

It follows from (8) and (9) that

$$|M_{56}| \leq \frac{C_3}{a_{56}} C_2 \left(|M_{25}| + |M_{26}| + |M_{45}| + |M_{46}| \right).$$

Therefore by (10) and (11),

$$
\begin{aligned}
(12) \quad |M_{56}| &\leq \frac{C_2 C_3}{a_{56}} \left(|M_{26}| + |M_{46}| + 2 \frac{a_{56} + C_3}{a_{56} - C_3} \left(|M_{15}| + |M_{35}| \right) \right) \\
&\leq \frac{C_2 C_3}{a_{56}} \cdot 2 \frac{a_{56} + C_3}{a_{56} - C_3} \left(|M_{26}| + |M_{46}| + |M_{15}| + |M_{35}| \right) \\
&\leq C_1 \left(|M_{26}| + |M_{46}| + |M_{15}| + |M_{35}| \right).
\end{aligned}
$$

Finally, we use a calibration-style calculation to obtain a contradictory estimate that $|M_{56}|$ is relatively large.

$$
\begin{aligned}
\text{cost}(N) &= \sum_{i<j\leq 4} a_{ij} |N_{ij}| = \sum_{1\leq i<j\leq 4} \text{flux}_{N_{ij}}(p_j - p_i) \\
&= \sum_{i=1}^{4} \text{flux}_{S_i}(p_i) = \sum_{i=1}^{6} \text{flux}_{S_i}(p_i) \\
&= \sum_{1\leq i<j\leq 6} \text{flux}_{M_{ij}}(p_j - p_i) \\
&\leq \sum_{1\leq i<j\leq 6} |p_i - p_j| \cdot |M_{ij}|.
\end{aligned}
$$

Since M was chosen to be cost-minimizing,

$$
\begin{aligned}
0 &> \text{cost}(M) - \text{cost}(N) \\
&\geq \sum_{1 \leq i < j \leq 6} a_{ij}|M_{ij}| - \sum_{1 \leq i < j \leq 6} |p_i - p_j| \cdot |M_{ij}| \\
&= \sum_{1 \leq i < j \leq 6} g(i,j)\, \epsilon\, |M_{ij}| \\
&= \big(|M_{26}| + |M_{46}| + |M_{15}| + |M_{35}|\big)C_1\epsilon - |M_{56}|\epsilon,
\end{aligned}
$$

contradicting Eq. (12). □

Remark 4.8. If instead we set

$$
g(i,j) = \begin{cases} -1 & \text{if } (i,j) = (5,6) \\ 0 & \text{otherwise,} \end{cases}
$$

then N still cannot be calibrated, but N is not minimizing for $F_1, F_2, \ldots, F_6$. So a non-calibrated cone for which placing points for the two absent fluids as close together as possible puts them on opposite vertices of their eyes (i.e., an exception to Theorem 4.4) may or may not be minimizing.

Remark 4.9. Kenneth Brakke ([**B1**], [**B2**], [**B3**]) has considered more general variable-coefficient calibrations than the constant-coefficient calibrations we use. He also considers more general networks (called real flat chains), where fluids are allowed to have fractional densities. We suspect that the network of our counterexample is not minimizing among these real-coefficient networks, and therefore cannot be calibrated even by variable-coefficient flows.

Remark 4.10. For further details on the proof of Theorem 4.6, as well as an alternate counterexample to Conjecture 2.5 whose proof is based on a more complicated calibration-style calculation that keeps track of angles in addition to lengths, see [**GNY**].

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