THE LOWEST VOLUME 3–ORBIFOLDS WITH HIGH TORSION

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ABSTRACT. For each natural number $n \geq 4$, we determine the unique lowest volume hyperbolic 3–orbifold whose torsion orders are bounded below by n. This lowest volume orbifold has base space the 3–sphere and singular locus the figure–8 knot, marked n. We apply this result to give sharp lower bounds on the volume of a hyperbolic manifold in terms of the order of elements in its symmetry group.

1. INTRODUCTION

The volume of a hyperbolic manifold is a fundamental and very useful topological invariant. Jørgensen and Thurston showed that the set of volumes of hyperbolic 3–manifolds forms a closed, non-discrete, well-ordered subset of $\mathbb{R}^+$ [30]. Dunbar and Meyerhoff proved that the set of volumes of hyperbolic 3–orbifolds — namely, quotients of manifolds by a discrete group action — has the same properties [13]. It follows that any family of hyperbolic manifolds or orbifolds contains a finite number of members realizing the smallest volume.

In the past decade, volume minimizers have been found for several important families. Gabai, Meyerhoff, and Milley [16, 27] proved that the smallest volume orientable hyperbolic 3–manifold is the Weeks manifold M_W, with $\mathrm{Vol}(M_W) \approx 0.9427$. Gehring, Marshall, and Martin [20, 25] have identified the smallest volume orientable 3–orbifold $\mathcal{O}_{\min}$, with $\mathrm{Vol}(\mathcal{O}_{\min}) \approx 0.03905$. There are also known results about non-compact manifolds and orbifolds [2, 4, 11, 26], fibered manifolds [1], and several other families [3, 5, 10].

In this paper, we identify the volume minimizers in infinitely many distinct families of orbifolds: namely, the family L_n of hyperbolic 3–orbifolds whose torsion orders are bounded below by n. For each $n \geq 4$, we prove that the unique smallest-volume orbifold in family L_n is the figure–8 knot marked n, as in Figure 1. See Theorem 1.2 below. In addition, we show that the volume of a hyperbolic 3–manifold M is bounded below by a constant times the size of its symmetry group G, where the constant depends on the orders of elements in G. These bounds are sharp in infinite families of examples.

1.1. Orbifolds.
A 3–dimensional, orientable *orbifold* $\mathcal{O}$ is locally modeled on B^3/G, where B^3 is the closed 3–ball and $G \subset SO(3)$ is a finite group acting on B^3 by rotations. The underlying topological space or *base space* is denoted $X_{\mathcal{O}}$. If a point in B^3 is fixed by a non-trivial element of G, the quotient of this point in $X_{\mathcal{O}}$ belongs to the *singular locus* $\Sigma_{\mathcal{O}}$.
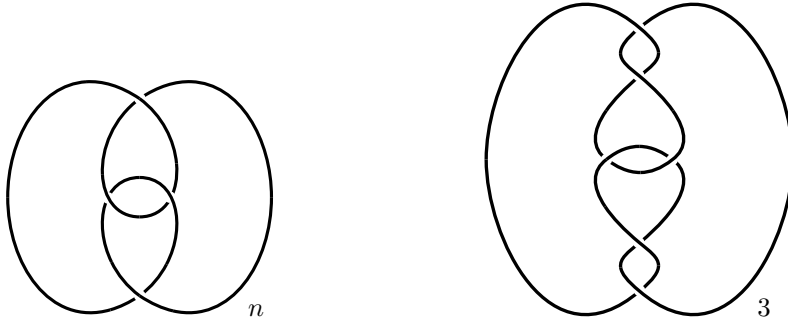
FIGURE 1. Left: the orbifold $\mathcal{P}_n$, with singular locus the figure–8 knot and torsion label n. By Theorem 1.2, $\mathcal{P}_n$ is the unique volume minimizer among hyperbolic orbifolds with torsion orders bounded below by $n \geq 4$. Right: the orbifold $\mathcal{O}_K$, with singular locus the 5_2 knot and torsion label 3.

All manifolds and all orbifolds appearing in this paper are assumed to be orientable. For a 3–orbifold $\mathcal{O}$, the base space $X_\mathcal{O}$ is a 3–manifold, possibly with boundary. The singular locus $\Sigma_\mathcal{O}$ is an embedded graph whose vertices in the interior of $X_\mathcal{O}$ have valence 3, and whose vertices on $\partial X_\mathcal{O}$ have valence 1. A regular neighborhood of a point in the interior of an edge of $\Sigma_\mathcal{O}$ is the quotient of B^3 under $\mathbb{Z}_n$, a finite cyclic group of rotations. The integer $n \geq 2$ is called the *torsion order* of the edge. A regular neighborhood of a trivalent vertex $v \in \Sigma_\mathcal{O}$ is the quotient of B^3 under an (orientation-preserving) spherical triangle group.

A 3–orbifold is called *hyperbolic* if $\mathcal{O} = \mathbb{H}^3/\Gamma$, where Γ is a discrete subgroup of $\mathrm{Isom}^+(\mathbb{H}^3)$, possibly with torsion. In parallel to the case where $\mathcal{O}$ is a manifold, the group Γ is called the *orbifold fundamental group of $\mathcal{O}$* and is denoted $\pi_1(\mathcal{O})$. For hyperbolic orbifolds, the singular locus $\Sigma_\mathcal{O}$ is the quotient of fixed points of torsion elements of Γ.

Definition 1.1. For $n \geq 2$, let L_n denote the set of all orientable hyperbolic 3–orbifolds with non-empty singular locus and with all torsion orders bounded below by n. Thus

$$L_2 \supset L_3 \supset L_4 \supset \ldots \quad \text{and} \quad \bigcap L_n = \emptyset.$$

Being a member of L_n imposes some additional structure. For instance, when $n \geq 4$ and $\mathcal{O} \in L_n$, Lemma 2.2 shows that the singular locus $\Sigma_\mathcal{O}$ is a closed 1–manifold; hence $\mathcal{O}$ is a *link orbifold* as in Definition 2.1.

The main result of this paper identifies the volume minimizer in L_n for each $n \geq 4$.

Theorem 1.2. *For all $n \geq 4$, the unique lowest–volume element of L_n is the orbifold $\mathcal{P}_n$, with base space S^3 and singular locus the figure–8 knot labeled n. See Figure 1, left.*

An exact formula for the volume of $\mathcal{P}_n$ was computed by Vesnin and Mednykh [31, Lemma 1]. See Table 1 in Section 3 for a table of approximate values. As $n \to \infty$, $\mathrm{Vol}(\mathcal{P}_n)$ converges to $2V_3 = 2.02988\ldots$, the volume of the figure–8 knot complement. See Section 5 for quantitative estimates on the rate of convergence.

The hypothesis that $n \geq 4$ is crucial for Theorem 1.2, since $\mathcal{P}_2$ is a spherical orbifold and $\mathcal{P}_3$ is Euclidean [12, Example 2.33]. Here is the state of knowledge about volume minimizers in L_n for these small values of n.

By the work of Gehring, Marshall, and Martin [20, 25], the unique volume minimizer of L_2 is the minimal–volume orbifold $\mathcal{O}_{\min}$, of volume $\mathrm{Vol}(\mathcal{O}_{\min}) = 0.03905\ldots$.

The volume minimizer in L_3 is currently unknown. In Corollary 2.6, we build on our previous work [10, Theorem 1.6] to show that $\mathrm{Vol}(\mathcal{O}) \geq 0.2371$ for every $\mathcal{O} \in L_3$. We conjecture that the unique volume minimizer in L_3 is the orbifold $\mathcal{O}_K$ depicted in Figure 1, right. Since $\mathrm{Vol}(\mathcal{O}_K) = \mathrm{Vol}(M_W)/3 = 0.3142\ldots$, there is currently a 25% gap between the conjectured and proven volume bounds.

1.2. **Symmetry groups.** One major reason to search for the smallest volume orbifold with a particular property is that any finite group G acting on a manifold M by isometries has an orbifold as a quotient. Thus universal lower bounds on the volume of M/G lead to upper bounds on the size of G. This connection is already present in dimension 2, where Hurwitz's famous $84(g-1)$ theorem about the maximal symmetry group of a genus g hyperbolic surface follows as a corollary of identifying $\pi/21$ as the minimal area of a hyperbolic 2–orbifold [23, 24]. In a similar vein, Gehring, Marshall, and Martin's identification of the smallest–volume orientable 3–orbifold $\mathcal{O}_{\min}$ implies that a hyperbolic 3–manifold M with symmetry group G must have volume at least $\mathrm{Vol}(\mathcal{O}_{\min})|G| > 0.039|G|$.

As a consequence of Theorem 1.2 and some intermediate results, we obtain a refinement of their result that depends on the orders of elements in G. Note that when $|G|$ is odd, the constant multiplying $|G|$ is much larger than 0.039.

Theorem 1.3. *Let M be an orientable hyperbolic 3–manifold of finite volume, and let G be a group of orientation-preserving isometries of M. Let p be the smallest prime number dividing $|G|$, or else 1 if G is trivial. Then*

$$\mathrm{Vol}(M) \geq |G| \cdot w_p, \quad \text{where} \quad w_p = \begin{cases} \mathrm{Vol}(\mathcal{O}_{\min}) \geq 0.03905, & p = 2, \\ 0.2371, & p = 3, \\ \mathrm{Vol}(\mathcal{P}_5) \geq 0.9372, & p = 5, \\ \mathrm{Vol}(M_W) \geq 0.9427, & otherwise. \end{cases}$$

See Theorem 6.2 in Section 6 for a slightly sharper version of Theorem 1.3, which takes into account the way in which G acts. See also Theorem 6.3 for stronger volume estimates under the additional hypothesis that M has cusps. That result strengthens and generalizes the work of Adams [3] and Futer, Kalfagianni, and Purcell [15] on cusped manifolds with symmetry.

1.3. **Organization.** We begin the proof of Theorem 1.2 by restricting the topology of $\mathcal{O}$. In Section 2, we show that the singular locus of any volume minimizer $\mathcal{O} \in L_n$ must be a link, with all components labeled n. In Section 3, we show that this link has a large embedded tube. When this tube is drilled out, the work of Agol and Dunfield [7], combined with that of Gabai, Meyerhoff, and Milley [16], implies that $\mathcal{O} \smallsetminus \Sigma_{\mathcal{O}}$ is one of only ten cusped hyperbolic manifolds. In other words, $\mathcal{O}$ must be obtained by *Dehn filling* one of these ten manifolds. See Theorem 3.7.

In Section 4, we prove Theorem 1.2 for $4 \leq n \leq 14$. We use a theorem of Futer, Kalfagianni, and Purcell [14] to limit the possibilities for $\mathcal{O}$ to finitely many Dehn

fillings of the manifolds from Theorem 3.7. Then, we rigorously check the finitely many fillings by computer, following Milley [27], to finish the proof.

In Section 5, we finish the proof of Theorem 1.2 for $n \geq 15$. For these values of n, Theorem 3.7 tells us that $\mathcal{O}$ is a Dehn filling of one of just *two* surgery parents, both of which have volume $2V_3 \approx 2.03$. Thus it suffices to estimate the change in volume during Dehn filling. We do this using the work of Hodgson and Kerckhoff on cone-manifolds [22].

Finally, in Section 6, we apply the above results to relate the volume of a hyperbolic manifold to the size of its symmetry group.

2. Link orbifolds

The goal of this section is to relate small-volume hyperbolic orbifolds to *link orbifolds*, which were the main object of study in our previous paper [10].

Definition 2.1. A 3–orbifold $\mathcal{O}$ is called a *link orbifold* if its singular locus $\Sigma_{\mathcal{O}}$ is a closed 1–manifold (i.e., a link) in $X_{\mathcal{O}}$.

We will show, in Proposition 2.5, that for any $n \geq 3$, a volume minimizer $\mathcal{O} \in L_n$ must be a link orbifold with n–torsion only. Combined with [10, Theorem 1.6], this will imply lower bounds on the volume of any orbifold $\mathcal{O} \in L_3$. In the next section, we use these results to further restrict the topology of any volume minimizer $\mathcal{O} \in L_n$.

Lemma 2.2. *Let $\mathcal{O}$ be a hyperbolic 3–orbifold with no 2–torsion. Then $\Sigma_{\mathcal{O}}$ has no vertices. Furthermore, either $\mathcal{O}$ is a link orbifold or it is non-compact, with an $S^2(3,3,3)$ cusp cross-section.*

Proof. Suppose, for a contradiction, that $\Sigma_{\mathcal{O}}$ contains a vertex v. Then the boundary of a regular neighborhood of v is a spherical 2–orbifold $S^2(p,q,r)$, where $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} > 1$. This cannot occur in our setting, where all torsion orders of $\mathcal{O}$ are bounded below by 3.

Next, suppose that a singular edge $e \subset \Sigma_{\mathcal{O}}$ is not a closed geodesic. Since $\mathcal{O}$ has no vertices, e must be a bi-infinite geodesic with both endpoints at a cusp. The cross-section of such a cusp is a Euclidean 2–orbifold, either $S^2(2,2,2,2)$ or $S^2(p,q,r)$ where $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} = 1$. The only possibility where $\mathcal{O}$ has no 2–torsion is $p = q = r = 3$. □

Lemma 2.3. *Let $\mathcal{O}$ be a hyperbolic 3–orbifold with no 2–torsion, and suppose $\mathcal{O}$ is not a link orbifold. Then $\mathcal{O}$ has 3–torsion, and $\mathrm{Vol}(\mathcal{O}) \geq V_3/2 = 0.50747\ldots,$ where V_3 is the volume of a regular ideal tetrahedron in $\mathbb{H}^3$.*

Proof. By Lemma 2.2, $\mathcal{O}$ is non-compact and has an $S^2(3,3,3)$ cusp cross-section. Orbifolds of this type were analyzed by Adams [4, Section 4]. He showed that there are exactly three hyperbolic 3–orbifolds with an $S^2(3,3,3)$ cusp cross-section and volume less than $V_3/2$. We call these exceptional orbifolds $\mathcal{O}_1, \mathcal{O}_2, \mathcal{O}_3$.

For each $\mathcal{O}_i$, let C_i be a horospherical neighborhood of an $S^2(3,3,3)$ cusp, which is *maximal* in the sense that no larger neighborhood is embedded. Lifting C_i to the universal cover $\mathbb{H}^3$ produces an arrangement of horoballs tangent at their boundaries, called a *horoball packing*. For each $\mathcal{O}_i$, Adams gives a complete description of the horoball packing of $\mathbb{H}^3$; see [4, Figure 3]. In particular, these three orbifolds share the feature that a fundamental domain for $\partial C_i \cong S^2(3,3,3)$ has a unique

point of self-tangency. As Adams notes [4, Lemma 2.1], this unique point of self-tangency of ∂C_i lifts to a point of tangency between two horoballs in $\mathbb{H}^3$, with an order–2 elliptic element of $\pi_1(\mathcal{O}_i)$ interchanging the two horoballs.

This cannot occur under the hypotheses of this lemma, as our orbifold $\mathcal{O}$ has no 2–torsion. Therefore, $\mathrm{Vol}(\mathcal{O}) \geq V_3/2$. □

Lemma 2.4. *Fix $n \geq 3$, and let $\mathcal{O} \in L_n$ be a hyperbolic link orbifold. If $n = 3$, suppose in addition that $\mathrm{Vol}(\mathcal{O}) < 0.4408$. Let $\mathcal{O}'_n$ be a link orbifold with the same base space and singular locus as $\mathcal{O}$, but with all torsion orders changed to n.*

Then $\mathcal{O}'_n$ is hyperbolic. Furthermore, $\mathrm{Vol}(\mathcal{O}'_n) \leq \mathrm{Vol}(\mathcal{O})$, with equality if and only if $\mathcal{O}'_n = \mathcal{O}$.

Proof. This is a restatement of [10, Proposition 2.8 and Lemma 5.1]. The hard part of the proof is showing that $\mathcal{O}'_n$ is hyperbolic; once this is established, the volume inequality follows from Schläfli's formula [12, Theorem 3.20]. □

Combining the last two results gives us a lot of control over volume minimizers in L_n.

Proposition 2.5. *Let $n \geq 3$, and let $\mathcal{O} \in L_n$ be a hyperbolic orbifold that minimizes volume among all orbifolds in L_n. Then*

   (1) *$\mathcal{O}$ is a link orbifold.*
   (2) *Every component of $\Sigma_{\mathcal{O}}$ has torsion order exactly n.*

Proof. If $n \geq 4$, conclusion (1) is a restatement of Lemma 2.2. If $n = 3$, and $\mathcal{O} \in L_3$ is not a link orbifold, Lemma 2.3 says that $\mathrm{Vol}(\mathcal{O}) > 0.5074$. This volume is not minimal in L_3, because there is an orbifold $\mathcal{O}_K \in L_3$, as in Figure 1 (right), such that $\mathrm{Vol}(\mathcal{O}_K) \approx 0.31423$. Thus a volume minimizer $\mathcal{O} \in L_n$ must be a link orbifold, proving (1).

Conclusion (2) is immediate from Lemma 2.4, because the orbifold $\mathcal{O}_K$ of Figure 1 shows that a volume minimizer in L_3 must have volume less than 0.44. □

Corollary 2.6. *Let $\mathcal{O} \in L_3$ be a hyperbolic orbifold. Then $\mathrm{Vol}(\mathcal{O}) \geq 0.2371$.*

Proof. By Proposition 2.5, it suffices to consider the case where $\mathcal{O} \in L_3$ is a link orbifold. Now, [10, Theorem 1.6] gives the estimate $\mathrm{Vol}(\mathcal{O}) \geq 0.2371$. □

3. TOPOLOGICAL REDUCTION

The goal of this section is to drastically restrict the topological and combinatorial possibilities for the volume minimizer in L_n for $n \geq 4$. Given any orbifold $\mathcal{O}$ that minimizes volume in L_n, we will show in Theorem 3.7 that the complement of the singular locus is one of ten topological types. This reduces the problem of finding the volume minimizer in L_n to a manageable Dehn surgery problem, which will be analyzed in Sections 4 and 5.

The proof of Theorem 3.7 proceeds in several steps.

   1. By Proposition 2.5, we already know that the singular locus $\Sigma_{\mathcal{O}}$ is a link with all components marked n.

   2. For any link orbifold $\mathcal{O} \in L_n$, there is a large embedded tube about $\Sigma_{\mathcal{O}}$. Explicit estimates on the tube radius (in terms of n) follow from results of Gehring, Maclachlan, Martin, and Reid [17, 19], and are derived in Proposition 3.2.

3. We drill out the singular locus of $\mathcal{O}$, obtaining a cusped hyperbolic 3–manifold $M_\mathcal{O} = X_\mathcal{O} \smallsetminus \Sigma_\mathcal{O}$. Given lower bounds on the radius of a tube about $\Sigma_\mathcal{O}$, a theorem of Agol and Dunfield [7] gives lower bounds on the ratio $\mathrm{Vol}(\mathcal{O})/\mathrm{Vol}(M_\mathcal{O})$. This bound, in terms of n, is derived in Proposition 3.5.

4. Gabai, Meyerhoff, and Milley have enumerated the ten cusped hyperbolic 3–manifolds of lowest volume [16]. Combining their enumeration with Proposition 3.5 and the knowledge that $\mathrm{Vol}(\mathcal{O}) \geq \mathrm{Vol}(\mathcal{P}_n)$ completes the proof of Theorem 3.7.

The first step of this program requires studying embedded tubes about the singular locus.

Definition 3.1. For $n \geq 4$, define a sequence

$$x_n = \begin{cases} 1/2 + \sqrt{3}/2 = 1.3660\ldots & \text{if } n = 4, \\ 1.84028 & \text{if } n = 5, \\ 2.41383 & \text{if } n = 6, \\ 2.65579 & \text{if } n = 7, \\ -1 + 1/2\sin^2(\pi/n) & \text{if } n \geq 8. \end{cases}$$

Proposition 3.2. *Let $\mathcal{O}$ be a finite–volume, orientable, hyperbolic link orbifold, all of whose torsion orders are equal to $n \geq 4$. Then the singular locus $\Sigma_\mathcal{O} \subset \mathcal{O}$ has an embedded tubular neighborhood of radius at least*

$$r_n = \mathrm{arccosh}(x_n)/2,$$

where x_n is as in Definition 3.1.

Proof. This is a consequence of the work of Gehring, Maclachlan, Martin, and Reid [17, 19] on axial distances in Kleinian groups. Let us describe the relevant results.

First, let $\mathcal{O} = \mathbb{H}^3/\Gamma$ be any (orientable) hyperbolic 3–orbifold. A singular geodesic $\alpha \subset \Sigma_\mathcal{O}$ with torsion label $n \geq 2$ lifts to a Γ–orbit of geodesic axes in $\mathbb{H}^3$, with each axis fixed pointwise by an elliptic subgroup $\mathbb{Z}_n \subset \Gamma$. When $n \geq 4$, Gehring and Martin [19] showed that two distinct axes of n–torsion must be separated by distance at least $b(n)$, where

$$\cosh b(n) = \begin{cases} 1 + \sqrt{5}/5 & \text{if } n = 5, \\ 2 & \text{if } n = 6, \\ -1 + 1/2\sin^2(\pi/7) & \text{if } n = 7, \\ x_n & \text{if } n = 4 \text{ or } n \geq 8. \end{cases}$$

The definition of $b(n)$ appears in [19, equations (4.16) and (4.17)], and the lower bound on axial distance is proved in Theorem 4.18 of that paper. See also [18, Section 4].

If all torsion subgroups of Γ are of the same order $n \geq 4$ and the elliptic axes are separated by distance at least $b(n)$, then tubes of radius $b(n)/2$ about these axes will project to disjointly embedded tubes about $\Sigma_\mathcal{O}$ in $\mathcal{O} = \mathbb{H}^3/\Gamma$. This proves the proposition for $n \neq 5, 6, 7$.

To complete the proof for these values of n, we need to relate orbifold groups to 2–generator Kleinian groups. Note that a geodesic $\alpha \subset \Sigma_\mathcal{O}$ is called *simple* if its lift to $\mathbb{H}^3$ is a disjoint union of hyperbolic lines, with disjoint endpoints at infinity.

In a link orbifold, all singular geodesics are simple. The following statement is [10, Lemma A.3]; see also [19, Lemma 2.26].

Lemma 3.3. *Let $\mathcal{O} = \mathbb{H}^3/\Gamma$ be a hyperbolic link orbifold, all of whose torsion orders are equal to $n \geq 3$. Then there is a Kleinian group $G = \langle f, g \rangle$, generated by an elliptic $f \in \Gamma$ of order n and an elliptic g of order 2, such that the axis of f is again simple in G, and such that the distance $\delta(f, g)$ between the axes of f and g is equal to the radius of a maximal tube about $\Sigma_{\mathcal{O}}$. Furthermore, there is an index–2 subgroup $H \subset G$ that is also a subgroup of Γ.*

Gehring, Maclachlan, Martin, and Reid have gone a long way toward classifying such 2–generator groups $G = \langle f, g \rangle$ for which the distance $\delta(f, g)$ is particularly small [17].

When $n = 5$, they show [17, Table 3] that if $\delta(f, g) < 0.6097$, then G must be conjugate to one of three groups $G_{5,i}$ for $i = 1, 2, 3$. By [17, Table 12], these three groups have the property that the axis of f is not simple. Thus, by Lemma 3.3, these three groups cannot occur in our setting when $\mathcal{O}$ is a link orbifold. Hence, the radius of an embedded tube about $\Sigma_{\mathcal{O}}$ is at least

$$\delta(f, g) \geq 0.6097 \geq \operatorname{arccosh}(1.84028)/2.$$

When $n = 6$, a similar argument applies. By [17, Table 4], we see that if $\delta(f, g) < 0.7642$, then G must be the group $G_{6,1}$ that realizes the minimal axial distance for $n = 6$. However, by [17, Table 12], the axis of f is not simple in this group. Thus, under our hypotheses, the radius of an embedded tube about $\Sigma_{\mathcal{O}}$ is at least

$$\delta(f, g) \geq 0.7642 \geq \operatorname{arccosh}(2.41383)/2.$$

When $n = 7$, [17, Table 5] shows that if $\delta(f, g) < 0.8162$, then G must be the group $G_{7,1}$ that realizes the minimal axial distance for $n = 7$. For this group, the distance $\delta(f, g)$ between axes f and g is exactly $b(7)/2$. As Gehring and Martin describe in [19, Example 8.11], this group $G_{7,1}$ is the $(2, 3, 7)$ triangle group, which contains 3–torsion in addition to 7–torsion. But then the index–2 subgroup $H \subset G$ would also contain 3–torsion, which is a contradiction because $H \subset \Gamma$ and all torsion orders of Γ are 7. Thus, under our hypotheses, the radius of an embedded tube about $\Sigma_{\mathcal{O}}$ is at least

$$\delta(f, g) \geq 0.8162 \geq \operatorname{arccosh}(2.65579)/2,$$

completing the proof. □

Remark 3.4. A modified version of the above argument applies for $n = 3$, with more special cases to consider. See [10, Theorem A.1] for the best available statement.

The argument now proceeds by considering the cusped manifold $M_{\mathcal{O}}$ obtained by drilling $\Sigma_{\mathcal{O}}$ out from $\mathcal{O}$. We use the following result to estimate the change in volume when drilling.

Proposition 3.5. *Let $\mathcal{O}$ be a finite–volume, orientable, hyperbolic link orbifold, all of whose torsion orders are equal to $n \geq 4$. Let $M_{\mathcal{O}} = X_{\mathcal{O}} \smallsetminus \Sigma_{\mathcal{O}}$ be the cusped hyperbolic manifold obtained by drilling out $\Sigma_{\mathcal{O}}$. Then*

$$\frac{\operatorname{Vol}(\mathcal{O})}{\operatorname{Vol}(M_{\mathcal{O}})} \geq \frac{(x_n^2 - 1)^{3/2}}{x_n^3} \left(1 + \frac{0.91}{x_n}\right)^{-1},$$

where x_n is as in Definition 3.1.

TABLE 1. Proposition 3.5 gives lower bounds on the volume of an orbifold $\mathcal{O} \in L_n$ if one assumes a lower bound on the volume of the cusped manifold $M_{\mathcal{O}} = \mathcal{O} \smallsetminus \Sigma_{\mathcal{O}}$. All values are truncated at 5 decimal places. Values listed in gray appear for completeness only and are not used in our arguments.

n	Lower bound if $\mathrm{Vol}(M_{\mathcal{O}}) \geq 2.568$	Lower bound if $\mathrm{Vol}(M_{\mathcal{O}}) \geq 2.848$	Lower bound if $\mathrm{Vol}(M_{\mathcal{O}}) \geq 3.6638$	$\mathrm{Vol}(\mathcal{P}_n)$
4	0.48729	0.54043	0.69524	0.50747
5	1.01654	1.12738	1.45034	0.93720
6	1.40605	1.55936	2.00606	1.22128
7	1.52065	1.68646	2.16958	1.41175
8	1.40625	1.55958	2.00635	1.54386
9	1.72961	1.92420	2.47542	1.63860
10	1.93362	2.15116	2.76740	1.70857
11	2.06917	2.30195	2.96139	1.76158
≥ 11	≥ 2.06917	≥ 2.30195	≥ 2.96139	< 2.02990

Proof. Proposition 3.2 gives a lower bound r_n on the radius of an embedded tubular neighborhood of $\Sigma_{\mathcal{O}}$, where

$$x_n = \cosh 2r_n.$$

Now, a theorem of Agol and Dunfield (see [7, Theorem 10.1] and [6, page 2299]) implies

$$\frac{\mathrm{Vol}(M_{\mathcal{O}})}{\mathrm{Vol}(\mathcal{O})} \leq (\coth^3 2r_n)\left(1 + \frac{0.91}{\cosh 2r_n}\right) = \left(\frac{x_n}{\sqrt{x_n^2 - 1}}\right)^3\left(1 + \frac{0.91}{x_n}\right).$$

See [10, Proposition 4.3] for a proof of the above estimate in the setting of link orbifolds. Taking reciprocals gives the desired estimate. $\qquad\square$

Theorem 3.6 (Gabai–Meyerhoff–Milley [16])**.** *Let N be a cusped orientable hyperbolic 3–manifold. If $\mathrm{Vol}(N) \leq 2.848$, then N has exactly one cusp and is one of the Snappea census manifolds* m003, m004, m006, m007, m009, m010, m011, m015, m016, *or* m017*.*

Furthermore, if $\mathrm{Vol}(N) \leq 2.568$, then $N =$ m003 *or* m004*, and $\mathrm{Vol}(N) = 2V_3 = 2.02988\ldots$.*

Proof. The enumeration of cusped manifolds of volume at most 2.848 is [16, Corollary 1.2]. The two volume minimizers on the list, m003 and m004, have volume $2V_3$ and were previously identified by Cao and Meyerhoff [11]. The next smallest volume is $\mathrm{Vol}(\text{m006}) \geq 2.5689$. $\qquad\square$

We can now prove the main result of this section.

Theorem 3.7. *Suppose that $n \geq 4$, and let $\mathcal{O} \in L_n$ be a volume minimizer in L_n. Then $M_{\mathcal{O}} = \mathcal{O} \smallsetminus \Sigma_{\mathcal{O}}$ is one of the Snappea census manifolds* m003, m004, m006, m007, m009, m010, m011, m015, m016, *or* m017*.*

Furthermore, if $n \neq 4, 8$, then $M_{\mathcal{O}} = \mathcal{O} \smallsetminus \Sigma_{\mathcal{O}}$ is one of m003 *or* m004*.*

Proof. By Proposition 2.5, $\mathcal{O}$ is a link orbifold, such that each component of $\Sigma_{\mathcal{O}}$ has torsion order exactly n. Now, Proposition 3.5 gives a lower bound on the ratio

$\text{Vol}(\mathcal{O})/\text{Vol}(M_\mathcal{O})$, and Theorem 3.6 gives a lower bound on $\text{Vol}(M_\mathcal{O})$. Multiplying these together gives a lower bound on $\text{Vol}(\mathcal{O})$; see Table 1.

When $n = 4$ or $n = 8$, we look for lower bounds for $\text{Vol}(\mathcal{O})$ under the assumption that $\text{Vol}(M_\mathcal{O}) \geq 2.848$. As Table 1 shows, the lower bound for $\text{Vol}(\mathcal{O})$ exceeds the volume of $\mathcal{P}_n$. Thus a volume minimizer in L_n must come from one of the ten census manifolds enumerated in Theorem 3.6.

When $n \geq 5$ and $n \neq 8$, we obtain sufficiently good lower bounds for $\text{Vol}(\mathcal{O})$ under the weaker assumption that $\text{Vol}(M_\mathcal{O}) \geq 2.568$. As Table 1 shows, the lower bound for $\text{Vol}(\mathcal{O})$ exceeds the volume of $\mathcal{P}_n$ when $5 \leq n \leq 11$ (except for $n = 8$). In fact, for $n = 11$, the lower bound for $\text{Vol}(\mathcal{O})$ exceeds the volume of the figure–8 knot complement m004, which is greater than $\text{Vol}(\mathcal{P}_n)$ for every n. For $n > 11$, the same lower bound on $\text{Vol}(\mathcal{O})$ applies, since by Lemma 2.4 the volume can only go down when we change the torsion labels of $\mathcal{O}$ to 11. Thus, if $\mathcal{O}$ is a volume minimizer in L_n for $n \geq 5$ and $n \neq 8$, Theorem 3.6 tells us that $M_\mathcal{O}$ is one of m003 or m004. □

We also obtain the following statement, which will be useful in Section 6.

Theorem 3.8. *Suppose that $n \geq 4$, and let $\mathcal{O} \in L_n$ be a link orbifold that is either non-compact or has disconnected singular locus. Then the volume of $\mathcal{O}$ is bounded below by the value in the next-to-last column of Table 1.*

Proof. We may assume without loss of generality that all torsion orders of $\mathcal{O}$ are equal to n. (Otherwise, replace $\mathcal{O}$ by $\mathcal{O}'_n$ as in Lemma 2.4, reducing volume in the process.)

Let $M_\mathcal{O}$ be the cusped 3–manifold obtained by drilling out $\Sigma_\mathcal{O}$. Under our hypotheses, $M_\mathcal{O}$ has at least two cusps; hence the work of Agol [5] implies that $\text{Vol}(M_\mathcal{O}) \geq V_8 = 3.6638\ldots$, where V_8 is the volume of a regular ideal octahedron. Now, we may apply Proposition 3.5, as in the last proof, and obtain the volume estimates in the next-to-last column of Table 1. □

4. The case $n \leq 14$: Computer analysis of Dehn fillings

Following Theorem 3.7, we know that for a volume minimizer $\mathcal{O} \in L_n$, the complement of the singular locus is one of ten cusped hyperbolic manifolds. In other words, $\mathcal{O}$ is obtained by *orbifold Dehn filling* one of these ten cusped manifolds. Let us review what this means.

Definition 4.1. Suppose that M is a hyperbolic 3–manifold with a torus cusp C. A *slope* on C is an unoriented, non-trivial homology class on ∂C. If we endow $H_1(\partial C) \cong \mathbb{Z}^2$ with a basis $\langle \mu, \lambda \rangle$, then we may write $s \in H_1(\partial C)$ as $\pm(a\mu + b\lambda)$. If a and b are relatively prime, then the Dehn filling $M(s)$ is the manifold resulting from attaching a non-singular solid torus to M, with the meridian disk mapped to s. If $(a, b) = n(a', b')$, where $n > 1$ and (a', b') is a primitive element of $\mathbb{Z}^2$, then $M(s)$ is a link orbifold obtained by attaching a singular solid torus to M, with the meridian disk mapped to $a'\mu + b'\lambda$, and with the core curve carrying the torsion label $n = \gcd(a, b)$.

To prove Theorem 1.2, it remains to find *which* Dehn filling of the cusped manifolds of Theorem 3.7 has the smallest volume. For any given value of n, we can restrict this to a finite search via the following theorem of Futer, Kalfagianni, and Purcell [14, Theorem 1.1]. We state it here for filling along a single cusp; see

[10, Theorem 4.5] for a general statement in the context of orbifolds with multiple cusps.

Theorem 4.2 (Futer–Kalfagianni–Purcell [14])**.** *Let M be a complete, finite-volume hyperbolic 3–manifold. Let C be an embedded horospherical cusp neighborhood in M, and let s be a slope on ∂C. If the length of s is $\ell(s) > 2\pi$, the Dehn filling $M(s)$ is hyperbolic, and*

$$\mathrm{Vol}(M(s)) \geq \left(1 - \left(\frac{2\pi}{\ell(s)} \right)^2 \right)^{3/2} \mathrm{Vol}(M).$$

We can now prove Theorem 1.2 in the case of $4 \leq n \leq 14$.

Theorem 4.3. *Let $\mathcal{O}$ be a volume minimizer in L_n, where $4 \leq n \leq 14$. Then $\mathcal{O} = \mathcal{P}_n$.*

Proof. Theorem 3.7 implies that $\mathcal{O}$ is a Dehn filling of M, where M belongs to the set

$$\mathcal{M} = \{\texttt{m003, m004, m006, m007, m009, m010, m011, m015, m016, m017}\}.$$

Fix a value B_n slightly larger than $\mathrm{Vol}(\mathcal{P}_n)$. Since $\mathcal{O}$ is a volume minimizer, we know that $\mathrm{Vol}(\mathcal{O}) < B_n$. Let $M \in \mathcal{M}$. Let $s = a\mu + b\lambda$ be a slope on ∂M. Since we are looking for slopes that produce orbifolds with n–torsion, we want $\gcd(a,b) = n$. Meanwhile, the constraint $\mathrm{Vol}(\mathcal{O}) < B_n$ places an upper bound on the length $\ell(a\mu + b\lambda)$. By Theorem 4.2, any filling slope s that gives a volume minimizer must be chosen from the set

$$\mathcal{F}_M^n(B_n) := \left\{ (a,b) \in \mathbb{Z}^2 \ : \ 1 - \left(\frac{2\pi}{\ell(a\mu + b\lambda)} \right)^2 \leq \left(\frac{B_n}{\mathrm{Vol}(M)} \right)^{2/3} \text{ and } \gcd(a,b) = n \right\}$$

where $M \in \mathcal{M}$. For each n, this gives a finite set of manifold–slope pairs.

For each n, this finite set of manifold–slope pairs can be rigorously analyzed by computer using a combination of Snap [21] and code developed by Moser [28] and Milley [27]. We can use Snap to compute approximate hyperbolic structures on the Dehn fillings in $\mathcal{F}_M^n(B_n)$, and the Milley–Moser code to certify that their volumes are all higher than that of $\mathcal{P}_n$ (except for $\mathcal{P}_n$ itself). See Milley [27] for a detailed description of the method.

For $n = 4$, the computer analysis appears in our previous work [10, Section 4]. We start by analyzing the case $n = 8$. Milley's code rigorously certifies that $\mathrm{Vol}(\mathcal{P}_8) < 1.55$; hence we may take $B_8 = 1.55$. The set

$$\bigcup_{M \in \mathcal{M}} \mathcal{F}_M^8(1.55)$$

consists of 42 slopes. For each of these fillings, Snap finds an approximate solution to the gluing equations in which all tetrahedra are positively oriented. For each such approximate solution, we use Milley's implementation of Moser's algorithm to rigorously verify that there is an actual positively oriented solution (i.e., an actual hyperbolic structure) near the approximate solution. This algorithm also gives an upper bound on the distance between each approximate solution and the actual solution. We then use Milley's code to rigorously verify that each filled orbifold has volume at least 1.55. The results of this verification show that indeed, each orbifold

under consideration except $\mathcal{P}_8$ has volume greater than 1.55, proving the theorem in the case where $n = 8$. The code is available in the ancillary files [9].

For all other values of n, Theorem 3.7 implies that it suffices to let M be one of m003 or m004. We let B_n equal the value of $\mathrm{Vol}(\mathcal{P}_n)$ rounded up to the nearest one-hundredth (see Table 1). Then

$$\mathcal{F}^n_{\mathtt{m003}}(B_n) = \{(-2n, n), (-n, n), (-n, 2n), (0, n), (n, 0), (n, n)\}$$

and

$$\mathcal{F}^n_{\mathtt{m004}}(B_n) = \{(-2n, n), (-n, n), (0, n), (n, 0), (n, n), (2n, n)\}.$$

For each of these Dehn fillings, we use Snap [21] to find a positively oriented approximate solution to the gluing equations and Milley's code to rigorously compute the volume. For each filling $\mathcal{O}$ except the orbifolds $\mathcal{P}_n$ and m003$(-10, 10)$, the Milley–Moser algorithm certifies that $\mathrm{Vol}(\mathcal{O}) > \mathrm{Vol}(\mathcal{P}_n)$. See the ancillary files [9].

For the initial triangulation of m003$(-10, 10)$ given by Snap, Moser's upper bound on distance between the approximate solution and the actual hyperbolic structure is too large for Milley's algorithm to certify that the volume is larger than B_{10}. Retriangulating and running the algorithm again certifies that its volume is also larger than B_{10}. See the files in the subfolder bad-ten of the ancillary files. □

The method of the above proof is likely to work for n in any bounded range, although the example of m003$(-10, 10)$ shows that the computer may need occasional hand-holding. However, to prove Theorem 1.2 in complete generality, we need to consider an infinite set of Dehn fillings of m003 and m004. This requires a different method.

5. The case $n \geq 15$: Analytic estimates on the change in volume

In this section, we complete the proof of Theorem 1.2 by showing that the orbifold $\mathcal{P}_n$ is the unique volume minimizer in the family L_n, for all $n \geq 4$. Following Theorem 4.3, it suffices to consider $n \geq 15$. Theorem 3.7 tells us that for $n \geq 9$, we only need to consider Dehn fillings of m003 and m004. Because $\mathrm{Vol}(\mathtt{m003}) = \mathrm{Vol}(\mathtt{m004}) = 2V_3$, it suffices to consider the *change* in volume when we perform Dehn filling.

Definition 5.1. Let M be a cusped hyperbolic 3–manifold with a horospherical cusp neighborhood C. For a slope s on ∂C, we define an *inverse norm*

$$A(s) = \frac{(2\pi)^2 \mathrm{Area}(\partial C)}{\ell(s)^2},$$

where $\ell(s)$ is the length of a geodesic representative of s on ∂C. Since expanding or shrinking the cusp C rescales the Euclidean metric on ∂C, it will leave $A(s)$ invariant. Note that $A(s) = (2\pi/L(s))^2$, where $L(s)$ is the normalized length of Hodgson and Kerckhoff [22].

For any Dehn filling slope s yielding a hyperbolic 3–manifold or orbifold $M(s)$, define

$$\Delta V(s) = \mathrm{Vol}(M) - \mathrm{Vol}(M(s)).$$

The last remaining step in the proof of Theorem 1.2 is the following result.

Proposition 5.2. *Fix $n \geq 15$. Then, among all Dehn fillings of* m003 *or* m004 *that produce link orbifolds with n–torsion, the largest value of ΔV is uniquely realized by* m004$(n, 0)$, *the filling that yields $\mathcal{P}_n$.*

Gromov and Thurston showed that $\Delta V(s)$ is always strictly positive [30, Theorem 6.5.6]. Neumann and Zagier [29] showed that $\Delta V(s)$ is well-predicted by the inverse norm of a slope:

$$\Delta V(s) = \frac{A(s)}{4} + O(A(s)^2).$$

However, since they do not quantify the error term $O(A(s)^2)$, their theorem is primarily an asymptotic estimate. Instead, we will prove Proposition 5.2 by applying the work of Hodgson and Kerckhoff on deformations of cone manifolds [22]. Their estimate on the change in volume $\Delta V(s)$ is in terms of the following functions:

$$l(z) = \frac{3.3957\, z^2 (z^2 - 3)(z^4 + 4z^2 - 1)}{2\,(z^4 - 6z^2 + 1)(z^2 + 1)^2},$$

$$u(z) = \frac{3.3957\, z^2 (z^4 + 4z^2 - 1)}{2\,(z^2 + 1)^3},$$

$$f(z) = \frac{3.3957(1 - z)}{\exp\left(\int_1^z F(w)\,dw\right)} \quad \text{where} \quad F(z) = \frac{-(z^4 + 6z^2 + 4z + 1)}{(z + 1)(z^2 + 1)^2},$$

$$\tilde{f}(z) = \frac{3.3957(1 - z)}{\exp\left(\int_1^z \tilde{F}(w)\,dw\right)} \quad \text{where} \quad \tilde{F}(z) = \frac{-(z^6 + 7z^4 + 12z^3 - 9z^2 - 4z + 1)}{(z + 1)(z^2 + 1)(z^2 + 2z - 1)(z^2 - 2z - 1)}.$$

Theorem 5.3 (Hodgson–Kerckhoff [22]). *Let M be a cusped hyperbolic 3–manifold. Let s be a slope on some cusp of M, satisfying $A(s) \leq f(1/\sqrt{3}) = 0.68653\ldots$. Then $M(s)$ is hyperbolic and*

$$\int_{\tilde{z}}^1 l(z)\,dz \leq \Delta V(s) \leq \int_{\hat{z}}^1 u(z)\,dz,$$

where $\hat{z} = f^{-1}(A(s))$ and $\tilde{z} = \tilde{f}^{-1}(A(s))$.

Proof. This is [22, Theorem 5.12], with simplified notation. The integrands in their statement of the theorem are expressed in terms of functions $G(z)$, $\tilde{G}(z)$, and $H(z)$ that are defined in [22, page 1079]. Those integrands simplify to $l(z)$ and $u(z)$, respectively. $\qquad\square$

Implicit in the statement of Theorem 5.3 is the claim that $f(z)$ and $\tilde{f}(z)$ are invertible in the right range. We verify this claim now.

Lemma 5.4. *The functions $f(z)$ and $\tilde{f}(z)$ are strictly decreasing on the interval $J = [z_0, 1]$, where $z_0 = \sqrt{-2 + \sqrt{5}} = 0.48586\ldots$. The inverse functions $f^{-1}(x)$ and $\tilde{f}^{-1}(x)$ are well-defined for $x \in [0, 0.699]$ and map the interval $[0, 0.699]$ into $[z_0, 1]$.*

Proof. We compute that

$$f'(z) = -2 \cdot 3.3957\, \frac{z(z^4 + 4z^2 - 1)}{(z + 1)(z^2 + 1)^2} \exp\left(\int_z^1 F(w)dw\right).$$

All the factors in the rational function are uniformly positive when $z > 0$, except for

$$p(z) = z^4 + 4z^2 - 1,$$

which has a zero at $z_0 = \sqrt{-2 + \sqrt{5}}$ and is positive thereafter. Thus $f'(z) < 0$ when $z \in (z_0, 1]$; hence f is monotonically decreasing and invertible on the closure of that domain. This means that $f^{-1}(x)$ is well-defined and decreasing on the interval

$$[f(1), \, f(z_0)] = [0, \, 0.69910 \dots].$$

The analysis for $\tilde{f}(z)$ is similar. We compute that

$$\tilde{f}'(z) = -2 \cdot 3.3957 \, \frac{z(z^2 - 3)(z^4 + 4z^2 - 1)}{(z + 1)(z^2 + 1)(z^2 + 2z - 1)(z^2 - 2z - 1)} \, \exp\left(\int_z^1 \tilde{F}(w) \, dw \right).$$

Among the factors of the rational function, the only ones that have a zero on $(0, 1]$ are $(z^2 + 2z - 1)$, with a zero at $\sqrt{2} - 1 < z_0$, and $p(z)$ with a zero at z_0. Thus $\tilde{f}'(z) < 0$ when $z \in (z_0, 1]$; hence $\tilde{f}^{-1}(x)$ is well-defined and decreasing on the interval

$$\left[\tilde{f}(1), \, \tilde{f}(z_0) \right] = [0, \, 1.6368 \dots]. \qquad \square$$

Definition 5.5. For a fixed n, let $\tilde{s} = \tilde{s}_n$ be the Dehn filling slope $(n, 0)$ on m004 that produces the orbifold $\mathcal{P}_n$. Let $s = s_n \neq \tilde{s}$ denote any other slope on a cusp of m003 or m004 that is n times a primitive slope. The subscript n will usually be implicit.

In this notation, Proposition 5.2 says that when $n \geq 15$,

$$\Delta V(\tilde{s}) > \Delta V(s).$$

We will derive Proposition 5.2 from Theorem 5.3, as follows.

1. We want to show that the lower bound on $\Delta V(\tilde{s})$ is larger than the upper bound on $\Delta V(s)$ for any potential competitor s. To that end, we show in Lemma 5.6 that the integrand $l(z)$ for the lower bound is larger than the integrand $u(z)$ for the upper bound.

2. We will show in Lemma 5.10 that the interval of integration for the lower bound contains the interval of integration for the upper bound. To do this, we will need
   (a) estimates on $A(s)$ and $A(\tilde{s})$, achieved in Lemma 5.7, and
   (b) an inequality relating the functions f and $\tilde{f}$, achieved in Lemma 5.9.

Lemma 5.6. *Let $z_0 \approx 0.48586$ be as in Lemma 5.4. Then $0 \leq u(z) \leq l(z)$ on the interval $J = [z_0, 1]$, with strict inequalities in the interior.*

Proof. Since

$$u(z) = \frac{3.3957 \, (z^4 + 4z^2 - 1)}{2 \, (1 + z^2)^3},$$

we have $u(z) > 0$ if and only if $p(z) = z^4 + 4z^2 - 1 > 0$. As we have seen in the proof of Lemma 5.4, we have $p(z) > 0$ whenever $z > z_0$. Thus $u(z) > 0$ on the interior of J.

Next, observe that

$$\frac{l(z)}{u(z)} = \frac{z^4 - 2z^2 - 3}{z^4 - 6z^2 + 1},$$

where the denominator $z^4 - 6z^2 + 1$ is negative on J. Therefore,

$$\frac{l(z)}{u(z)} > 1 \iff z^4 - 2z^2 - 3 < z^4 - 6z^2 + 1 \iff 4z^2 < 4,$$

which holds on the interior of J. □

Lemma 5.7. *Let s and $\tilde{s}$ be as in Definition 5.5. Then*

$$A(\tilde{s}) = 8\sqrt{3}\pi^2/n^2 \quad and \quad A(s) \leq 2\sqrt{3}\pi^2/n^2.$$

In particular, $4A(s) \leq A(\tilde{s})$.

Proof. This can be easily computed using Snap [21]. In both manifolds m003 and m004, the maximal cusp C has Area$(\partial C) = 2\sqrt{3}$.

For m004, the shortest simple closed curve is $(1,0)$, the knot-theoretic meridian, of length 1. Thus the slope $\tilde{s} = (n,0)$ has length n; hence by Definition 5.1, $A(\tilde{s}) = 8\sqrt{3}\pi^2/n^2$.

The shortest primitive slope on a maximal cusp of m003 has length 2, while the second-shortest primitive slope on a maximal cusp of m004 is the $(0,1)$ slope of length $2\sqrt{3}$. Thus, if s is n times a primitive slope and $s \neq \tilde{s}$, we have $\ell(s) \geq 2n$. Therefore, by Definition 5.1, $A(s) \leq 2\sqrt{3}\pi^2/n^2$. □

Lemma 5.8. *Let $n \geq 15$. Then both $\tilde{s} = \tilde{s}_n$ and $s = s_n$ satisfy the hypotheses of Theorem 5.3. In addition, both $\tilde{z} = \tilde{f}^{-1}(A(\tilde{s}))$ and $\hat{z} = f^{-1}(A(s))$ lie in the interval $I = [0.83, 1]$.*

Proof. By Lemma 5.7, we know that when $n \geq 15$, the inverse norms of s and $\tilde{s}$ satisfy

$$0 < 4A(s) \leq A(\tilde{s}) \leq \frac{8\sqrt{3}\pi^2}{15^2} \approx 0.60781.$$

This means that both s and $\tilde{s}$ satisfy hypotheses of Theorem 5.3. By Lemma 5.4, it also means that the values $\hat{z} = f^{-1}(A(s))$ and $\tilde{z} = \tilde{f}^{-1}(A(\tilde{s}))$ are well-defined. Since f^{-1} and $\tilde{f}^{-1}$ are monotonically decreasing by Lemma 5.4, we know that

$$\tilde{z} = \tilde{f}^{-1}(A(\tilde{s})) \geq \tilde{f}^{-1}\left(\frac{8\sqrt{3}\pi^2}{15^2}\right) = 0.83211\ldots$$

and

$$\hat{z} = f^{-1}(A(s)) \geq f^{-1}\left(\frac{2\sqrt{3}\pi^2}{15^2}\right) = 0.95182\ldots. \qquad □$$

Lemma 5.9. *On the interval $I = [0.83, 1]$, we have $\tilde{f}(z) \leq 4f(z)$, with a strict inequality in the interior of I.*

Proof. Note that $\tilde{f}(1) = 4f(1) = 0$, while

$$\tilde{f}(0.83) = 0.61577\ldots < 1.75736\ldots = 4f(0.83).$$

Thus the desired inequality is true at the endpoints of I. To prove that $\tilde{f}(z) < 4f(z)$ in the interior of I, we show that $4f$ is concave down on I (hence above its secant line), while $\tilde{f}$ is concave up (hence below its secant line).

In Lemma 5.4, we computed f' and $\tilde{f}'$. Now, we differentiate again:

$$f''(z) = -2 \cdot 3.3957 \frac{z^8 + 6z^6 + 32z^4 + 10z^2 - 1}{(z^2 + 1)^4(z + 1)} \exp\left(\int_z^1 F(w)\, dw\right).$$

All the factors in the denominator are positive. The polynomial

$$g(z) = z^8 + 6z^6 + 32z^4 + 10z^2 - 1$$

is increasing on I, and satisfies $g(0.83) > 20$. Hence $g(z) > 0$ on I, and we conclude that $f''(z) < 0$ on I.

Next, we compute that

$\tilde{f}''(z)$

$$= -2 \cdot 3.3957 \frac{z^{12} - 4z^{10} + 17z^8 - 248z^6 + 203z^4 - 36z^2 + 3}{(z+1)(z^2+1)^2(z^2+2z-1)^2(z^2-2z-1)^2} \exp\left(\int_z^1 \tilde{F}(w)\,dw\right).$$

Since all factors in the denominator are positive, it follows that $\tilde{f}''(z) > 0$ on I if and only if

$$q(z) = z^{12} - 4z^{10} + 17z^8 - 248z^6 + 203z^4 - 36z^2 + 3 < 0$$

on I. We check this by analyzing $r(z) = q(\sqrt{z})$.

Claim. The function $r(z) = z^6 - 4z^5 + 17z^4 - 248z^3 + 203z^2 - 36z + 3$ is strictly negative on the interval $[0.83^2, 1]$.

We prove this claim by "downward induction" on the derivatives of r. In fact, we will show that for $k \in \{1, 2, 3, 4\}$, $r^{(k)}(z)$ has no zeros on the larger interval $[0.5, 1]$.

First, check that $r^{(4)}$, a quadratic function, is always positive on $[0.5, 1]$. This implies that $r^{(3)}$ is increasing on $[0.5, 1]$. It follows from the fact that $r^{(3)}(1) < 0$ that $r^{(3)}$ is negative on the interval $[0.5, 1]$; hence r'' is decreasing on $[0.5, 1]$. Since $r''(0.5) < 0$, r'' is negative on $[0.5, 1]$; hence r' is decreasing. Since $r'(0.5) < 0$, r' is negative on $[0.5, 1]$. Therefore, r is decreasing on $[0.5, 1]$. Since $r(0.83^2) \approx -3.226 < 0$, this proves the claim.

The claim shows that $\tilde{f}''(z) > 0$ on I. Since we have already seen that $4f''(z) < 0$ on I, the proof is complete. □

Lemma 5.10. *Let $n \geq 15$, and let s and $\tilde{s}$ be slopes as in Definition 5.5. Let $\tilde{z} = \tilde{f}^{-1}(A(\tilde{s}))$ and $\hat{z} = f^{-1}(A(s))$. Then $\tilde{z} < \hat{z}$.*

Proof. When $n \geq 15$, Lemma 5.8 says that $\tilde{z}$ and $\hat{z}$ lie in the interval $I = [0.83, 1]$. Therefore,

$$4A(s) \leq A(\tilde{s}) = \tilde{f}(\tilde{z}) < 4f(\tilde{z}),$$

where the first inequality is by Lemma 5.7 and the second inequality is by Lemma 5.9. Since f^{-1} is a decreasing function and $A(s) < f(\tilde{z})$, we conclude that

$$\hat{z} = f^{-1}(A(s)) > \tilde{z}. \qquad \square$$

Finally, we can prove Proposition 5.2, the main result of this section.

Proof of Proposition 5.2. As in Definition 5.5, let $\tilde{s}$ be the slope on a cusp of m004 that produces the orbifold $\mathcal{P}_n$, and let $s \neq \tilde{s}$ be any other slope on m003 or m004 that is n times a primitive slope. Now,

$$\Delta V(s) \leq \int_{\hat{z}}^1 u(z)\,dz < \int_{\tilde{z}}^1 u(z)\,dz < \int_{\tilde{z}}^1 l(z)\,dz \leq \Delta V(\tilde{s}),$$

where the first and last inequalities are Theorem 5.3, the second inequality is Lemma 5.10, and the third inequality is Lemma 5.6. □

We can also complete the proof of Theorem 1.2.

Proof of Theorem 1.2. Let $\mathcal{O}$ be a volume minimizer in L_n, where $n \geq 4$. If $4 \leq n \leq 14$, Theorem 4.3 proves that $\mathcal{O} = \mathcal{P}_n$. If $n \geq 15$, Theorem 3.7 shows that $\mathcal{O}$ is a Dehn filling of m003 or m004. Now, Proposition 5.2 shows that among those Dehn fillings that produce n–torsion, $\mathcal{P}_n$ is the unique one of lowest volume. $\square$

6. SYMMETRY GROUPS OF MANIFOLDS

In this section, we apply the prior results of this paper (Theorem 1.2, Corollary 2.6, and Theorem 3.8) to relate the volume of a hyperbolic manifold to the size of its symmetry group. Recall, from the introduction, that bounds of this type were one of the main motivations driving the search for the lowest volume orbifold, compact and non-compact [20, 25, 26].

Definition 6.1. Let G be a group of isometries of a finite-volume hyperbolic 3–manifold M. It is a well-known consequence of Mostow rigidity that G is finite [30, Corollary 5.7.4]. We define the following numerical invariants of G:

- $n = n(G)$ is the smallest order of a non-trivial element of G stabilizing a point of M, or else 1 if G acts freely.
- $p = p(G)$ is the smallest prime number dividing $|G|$, or else 1 if G is trivial.

We emphasize that $n(G)$ depends on how G acts on M, whereas $p(G)$ depends only on $|G|$.

The following theorem is a generalization of Theorem 1.3, stated in the introduction.

Theorem 6.2. *Let M be an orientable hyperbolic 3–manifold of finite volume, and let G be a group of orientation-preserving isometries of M. Let n and p be as in Definition 6.1. Then*

$$\mathrm{Vol}(M) \geq |G| \cdot w_n \geq |G| \cdot w_p, \quad where \quad w_i = \begin{cases} \mathrm{Vol}(\mathcal{O}_{\min}) \geq 0.03905, & i = 2, \\ 0.2371, & i = 3, \\ \mathrm{Vol}(\mathcal{P}_4) \geq 0.5074, & i = 4, \\ \mathrm{Vol}(\mathcal{P}_5) \geq 0.9372, & i = 5, \\ \mathrm{Vol}(M_W) \geq 0.9427, & otherwise. \end{cases}$$

Furthermore, the above estimate is sharp for infinitely many pairs (M, G).

Proof. We consider two cases, depending on whether G acts freely or with fixed points.

If G acts freely, then $n = 1$. The quotient is $M/G = N$, an orientable hyperbolic 3–manifold. Thus, by the work of Gabai, Meyerhoff, and Milley [16, 27],

$$\mathrm{Vol}(M) = |G| \cdot \mathrm{Vol}(N) \geq |G| \cdot \mathrm{Vol}(M_W) = |G| \cdot w_1 \geq |G| \cdot w_p$$

for any p.

Otherwise, if the action of G has fixed points, we have $n \geq p \geq 2$, hence $w_n \geq w_p$. The quotient is an orbifold $M/G = \mathcal{O} \in L_n$. Combining Theorem 1.2, Corollary 2.6, and the work of Gehring, Marshall, and Martin [20, 25] gives

$$\mathrm{Vol}(M) = |G| \cdot \mathrm{Vol}(\mathcal{O}) \geq \begin{cases} |G| \cdot \mathrm{Vol}(\mathcal{O}_{\min}), & n = 2, \\ |G| \cdot 0.2371, & n = 3, \\ |G| \cdot \mathrm{Vol}(\mathcal{P}_n), & otherwise. \end{cases}$$

Since $\mathrm{Vol}(\mathcal{P}_n) > \mathrm{Vol}(M_W)$ for $n > 5$, we conclude that

$$\mathrm{Vol}(M) \geq |G| \cdot w_n \geq |G| \cdot w_p.$$

To see the sharpness of this estimate, let X be one of $\mathcal{O}_{\min}$, $\mathcal{P}_4$, $\mathcal{P}_5$, or M_W. Since Kleinian groups are residually finite [8], there are infinitely many epimorphisms $\varphi : \pi_1 X \to G$, with finite image and torsion-free kernel. For each such φ, we can let M be the cover of X corresponding to $\ker \varphi$. The theorem will then be sharp for (M, G). $\square$

There are similar (and sharper) lower bounds if one assumes that the manifold M has cusps. Here, Meyerhoff [26] and Adams [4] have identified the first several non-compact 3–orbifolds of lowest volume, and Adams has also identified the smallest volume 3–orbifolds that could occur as the quotient of a one-cusped manifold [3]. Futer, Kalfagianni, and Purcell have found lower bounds on the volume of a symmetric link complement in S^3, provided the symmetry group is cyclic [15]. The following result strengthens and generalizes those estimates.

Theorem 6.3. *Let M be a cusped orientable hyperbolic 3–manifold, and let G be any group of orientation-preserving isometries of M. Let n and p be as in Definition* 6.1*. Then*

$$\mathrm{Vol}(M) \;\geq\; |G| \cdot u_n \;\geq\; |G| \cdot u_p, \quad \text{where} \quad u_i = \begin{cases} V_3/12 \geq 0.08457, & i = 2, \\ V_3/2 \geq 0.50747, & i = 3, \\ 0.69524, & i = 4, \\ 1.45034, & i = 5, \\ 2.00606, & i = 6, \\ 2V_3 \geq 2.02988, & otherwise. \end{cases}$$

Furthermore, if M has a single cusp, then $\mathrm{Vol}(M) \geq |G| \cdot 0.30532\ldots$, where the decimal in the equation is $1/12$ the volume of a regular ideal octahedron.

Proof. As in the proof of Theorem 6.2, we consider two cases, depending on how G acts.

If G acts freely, then $n = 1$, hence $u_1 \geq u_p$ for any p. The quotient $N = M/G$ is a cusped orientable hyperbolic manifold. By the work of Cao and Meyerhoff [11], $\mathrm{Vol}(N) \geq 2V_3$.

If G acts with fixed points, then $n \geq p \geq 2$, hence $u_n \geq u_p$. The quotient M/G is a non-compact hyperbolic orbifold $\mathcal{O} \in L_n$, whose volume can be estimated as follows. If $n = 2$, then $\mathrm{Vol}(\mathcal{O}) \geq V_3/12$ by the work of Meyerhoff [26] and Adams [4]. If $n = 3$, then a cusp cross-section of $\mathcal{O}$ is either a torus or $S^2(3,3,3)$; in either case we have the estimate $\mathrm{Vol}(\mathcal{O}) \geq V_3/2$, by [10, Lemma 2.2] and Lemma 2.3 of this paper. If $4 \leq n \leq 6$, we turn to Theorem 3.8 and obtain volume estimates from Table 1.

When $n \geq 7$, any orbifold $\mathcal{O} \in L_n$ is also an element of L_7. By Table 1, the volume estimate for a non-compact orbifold $\mathcal{O} \in L_7$ is larger than $2.1 > 2V_3 \approx 2.03$. Therefore, for $n \geq 7$, the smaller volume bound comes from the case where M/G is a manifold.

The "furthermore" statement is a theorem of Adams [3, Corollary 8.3]. $\square$

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