# On diagrammatic bounds of knot volumes and spectral invariants

**David Futer · Efstratia Kalfagianni · Jessica S. Purcell**

**Abstract**    In recent years, several families of hyperbolic knots have been shown to have both volume and $\lambda_1$ (first eigenvalue of the Laplacian) bounded in terms of the twist number of a diagram, while other families of knots have volume bounded by a generalized twist number. We show that for general knots, neither the twist number nor the generalized twist number of a diagram can provide two-sided bounds on either the volume or $\lambda_1$. We do so by studying the geometry of a family of hyperbolic knots that we call *double coil knots*, and finding two-sided bounds in terms of the knot diagrams on both the volume and on $\lambda_1$. We also extend a result of Lackenby to show that a collection of double coil knot complements forms an *expanding family* iff their volume is bounded.

**Keywords**    Knot volume · Twist number · Generalized twist number · Expanding family

**Mathematics Subject Classification (2000)**    57M25 · 57M50 · 58J50

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## 1 Introduction

For any diagram of a knot, there is an associated 3-manifold: the complement of the knot in the 3-sphere. In the 1980s, Thurston proved that the complement of any non-torus, non-satellite knot admits a hyperbolic metric [4], which is necessarily unique up to isometry. As a result, geometric information about a knot complement, such as volume and the spectrum of the Laplacian, gives topological knot invariants. However, in practice, these invariants have been difficult to estimate with only a diagram of a knot.

Recently, there has been some progress in estimating geometric information from particular classes of diagrams. For volumes, Lackenby showed that the volume of an alternating knot complement is bounded above and below in terms of the twist number of an alternating diagram [20] (see Definition 1.1). We extended these results to highly twisted knots [11] and to sums of alternating tangles [12]. Purcell used a generalization of the twist number to find lower volume bounds for additional classes of knots [23], while in [10] we showed that the volume of a closed 3-braid is bounded above and below in terms of the generalized twist number of the braid. More recently, Lackenby showed that for alternating and highly twisted knots, the first eigenvalue $\lambda_1$ of the Laplacian can be estimated in terms of the twist number [18]. Based on these examples, one might hope that a suitable generalization of the twist number of a diagram controls the geometry of all hyperbolic knot complements.

In this paper, we show that the twisting in a diagram cannot give two-sided geometric bounds for general knots. We do so by presenting a class of knots, called *double coil knots*, for which the volume can be made bounded while the twist number becomes arbitrarily large, or the volume can be made unbounded while the generalized twist number stays constant. Similarly, we show that $\lambda_1$ can stay bounded while the twist number becomes arbitrarily large, or $\lambda_1$ can approach 0 while the generalized twist number stays constant.

To state our results more precisely, we need a few definitions.

**Definition 1.1** A diagram of a knot is a 4-valent graph with over-under crossing information at each vertex. A *twist region* of a diagram is a portion of the diagram consisting of bigons arranged end to end, which is maximal in the sense that there are no additional bigons adjacent to either end. A single crossing not adjacent to any bigons is also a twist region. See Fig. 1, left. Note that a twist region containing $c$ crossings corresponds to $c/2$ full twists of the two strands.

The number of twist regions in a particular diagram $D$ is called the *twist number of $D$*. The minimum of the twist numbers of $D$ as $D$ ranges over all diagrams of a knot $K$ is defined to be the *twist number* of $K$, and is denoted $\tau(K)$.

**Definition 1.2** A *generalized twist region* on $q$ strands, $q \geq 2$, is a region of a knot diagram consisting of $q$ strands twisted maximally. That is, if the $q - 2$ innermost strands are removed from a generalized twist region on $q$ strands, then the remaining two strands form a twist region as in Definition 1.1. These two outermost strands bound a twisted, rectangular ribbon. The additional $q - 2$ strands are required to run parallel to the two outermost strands,



**Fig. 1** *Left* a twist region. Two strands twist about each other maximally. *Right* a generalized twist region with two full twists

**Fig. 2** *Left* a $(1, 2)$ double coil knot. *Right* a $(3, 5)$ double coil knot
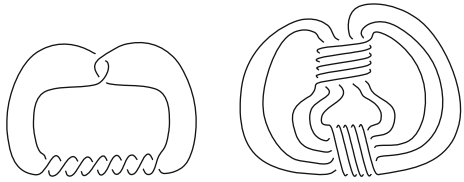
embedded on this ribbon. By definition, a twist region is a generalized twist on $q = 2$ strands. See Fig. 1, right.

In a given diagram $D$, there is typically more than one way to partition the crossings of $D$ into generalized twist regions. For example, a single generalized twist region can contain many ordinary twist regions. The *generalized twist number of D* is defined to be the smallest number of generalized twist regions, minimized over all partitions of $D$ into generalized twist regions.

**Definition 1.3** A *double coil knot* is a knot with exactly two generalized twist regions, where each twist region contains $q \geq 2$ strands and an integral number of full twists. At each end of each generalized twist region, $p < q$ strands split off to the right, while $q - p$ strands split off to the left. A knot $K$ with this description is called a $(p, q)$ *double coil knot*. Note that $K$ will be a knot precisely when $p$ and $q$ are relatively prime.

The integers $p$ and $q$, together with the number of full twists in each generalized twist region, completely specify a diagram of a double coil knot. See Fig. 2 for two examples.

Note that when $q = 2$ and one of the two generalized twist regions contains exactly one full twist, then corresponding double coil knot is a twist knot. See Fig. 2, left. Thus, double coil knots can be seen as a generalization of twist knots.

Every double coil knot is also a special case of a *double torus knot*: it can be embedded on an unknotted genus-2 surface in $S^3$. To visualize this genus-2 surface, start with the sphere obtained by compactifying the projection plane, and add one handle for each generalized twist region. Then, in each region of Fig. 2, the coils run around the cylinder of the handle. The family of double torus knots has been studied extensively (see e.g., [15–17]).

In Sect. 2, we prove the following two-sided, combinatorial estimate on the volumes of double coil knots.

**Theorem 2.9** *Let $p$ and $q$ be relatively prime integers with $0 < p < q$, and let $k$ be the length of the continued fraction expansion of $p/q$. Let $K$ be a $(p, q)$ double coil knot, in which each generalized twist region has at least 4 full twists. Then $K$ is hyperbolic, and*

$$0.9718\, k - 0.3241 \ \leq \ \mathrm{vol}(S^3 \backslash K) < 4v_8 k,$$

*where $v_8 = 3.6638\ldots$ is the volume of a regular ideal octahedron in $\mathbb{H}^3$.*

The length of the continued fraction expansion of $p/q$ turns out to be unrelated to either the twist number or the generalized twist number of a $(p, q)$ double coil knot. As a result, we can show that neither of those quantities predicts the volume of $K$.

**Theorem 3.3** *The volumes of hyperbolic double coil knots are not effectively predicted by either the twist number or generalized twist number. More precisely:*

(a) *For any $q \geq 3$, and any $p$ relatively prime to $q$, there exists a sequence $K_n$ of $(p, q)$ double coil knots such that $\tau(K_n) \to \infty$ while $\mathrm{vol}(S^3 \backslash K_n)$ stays bounded.*

*(b)   All double coils have generalized twist number* 2, *but their volumes are unbounded.*

Theorem 3.3 implies that the known upper bounds on volume in terms of twist number can be quite ineffective. Lackenby initially found an upper bound on volume that was linear in terms of twist number [20]. Agol and Thurston improved the constants in Lackenby's estimate, and showed that the upper bound is asymptotically sharp for a particular family of alternating links [20, Appendix]. However, for the double coil knots of Theorem 3.3, the volume is bounded but the estimate in terms of twist number will become arbitrarily large. This phenomenon occurs in much greater generality: see Theorem 3.1 for the most general statement, and Corollary 3.2 for an application to *m*-braids.

For a Riemannian manifold $M$, $\lambda_1(M)$ is defined to be the smallest positive eigenvalue of the Laplace–Beltrami operator $\Delta f = -\mathrm{div\ grad\ } f$. It turns out that the volume and $\lambda_1$ of a double coil knot are closely related. In Sect. 4, we show the following result.

**Theorem 4.3** *Let K be a hyperbolic double coil knot. Then*

$$\frac{A_1}{\mathrm{vol}(S^3 \backslash K)^2} \ \le \ \lambda_1(S^3 \backslash K) \ \le \ \frac{A_2}{\mathrm{vol}(S^3 \backslash K)},$$

*where $A_1 \ge 8.76 \times 10^{-15}$ and $A_2 \le 12650$.*

Combining Theorem 3.3 with Theorem 4.3 immediately gives the following.

**Corollary 4.5** *The first eigenvalue of the Laplacian of hyperbolic double coil knots is not effectively predicted by either the twist number or generalized twist number. More precisely:*

(a)   *For any $q \ge 3$, and any $p$ relatively prime to $q$, there exists a sequence $K_n$ of $(p, q)$ double coils such that $\tau(K_n) \to \infty$ while $\lambda_1(S^3 \backslash K_n)$ is bounded away from 0 and $\infty$.*
(b)   *All double coil knots have generalized twist number* 2, *but the infimum of $\{\lambda_1(S^3 \backslash K_n)\}$ is zero.*

Theorem 4.3 also extends a result of Lackenby about expanding families. Recall that a collection $\{M_i\}$ of Riemannian manifolds is called an *expanding family* if $\inf \lambda_1(M_i) > 0$. Lackenby showed that knots whose volumes are bounded above and below by the twist number form an expanding family if and only if their volumes are bounded [18, Theorem 1.7]. Even though the volumes of double coil knots are very far from being governed by the twist number, Theorem 4.3 implies that a sequence of double coil knots forms an expanding family if and only if their volumes are bounded.

This paper is organized as follows. In Sect. 2, we study the geometry and combinatorics of a certain surgery parent of double coil knots. The volume estimates for these parent links lead to volume estimates for double coil knots in Theorem 2.9. In Sect. 3, we construct hyperbolic knots that have bounded volume but arbitrarily large twist number. In Sect. 4, we describe the connection between the volume of a double coil knot and its first eigenvalue $\lambda_1$. Our main tool here is Theorem 4.1, which gives two-sided bounds for $\lambda_1$ of a finite-volume hyperbolic 3-manifold, in terms of the volume and the Heegaard genus.

## 2 Volume estimates for double coil knots

In this section, we study the volumes of double coil knots. We begin by showing that a $(p, q)$ double coil knot is obtained by Dehn filling a certain 3-component link, closely related to the 2-bridge link of slope $p/q$. Next, we obtain two sided diagrammatic bounds on volumes

of these parent links. Finally, we apply a result of the authors [11] to bound the change in volume under Dehn filling, obtaining two-sided diagrammatic estimates on the volume of the double coil knots.

## 2.1 Augmentations of double coil knots

A twist knot as in Fig. 2a may be viewed as a Dehn filling of the Whitehead link, which is itself a Dehn filling of the Borromean rings. Similarly, we may view double coil knots as Dehn fillings of a class of link complements in $S^3$. The idea is as follows. At each of the two generalized twist regions of a double coil knot, insert a *crossing circle* $C_i$, namely a simple closed curve encircling all $q$ strands of the generalized twist. The complement of the resulting three-component link is homeomorphic to the complement of the three-component link with all full twists removed from each twist region. Examples of such links are shown in Fig. 3. We call such a link the *augmentation* of a double coil knot.

The augmentation of a $(p, q)$ double coil knot has a simple description in terms of the rational number $p/q$, as follows. The augmentation consists of three components. Two, namely $C_1$ and $C_2$, can be isotoped to lie orthogonal to the projection plane, bounding disjoint disks $D_1$ and $D_2$ in $S^3$. The third component can be isotoped to be a nontrivial simple closed curve embedded on the projection plane, disjoint from the intersections of $C_1$ and $C_2$ with the projection plane. We adopt the convention that the projection plane contains a point at infinity, forming a sphere in $S^3$. Note that the projection sphere minus the four points of intersection with $C_1$ and $C_2$ is a 4-punctured sphere $S$. Once we have determined a framing for $S$, any simple closed curve can be described by a number in $\mathbb{Q} \cup \{1/0\}$.

We choose our framing as follows. Let $1/0 = \infty$ be the simple closed curve on $S$ that is disjoint from $D_1$ and $D_2$, and separates those disks from each other. Now, draw a straight arc $A$ connecting one of the punctures of $C_1$ with one of $C_2$, as in Fig. 4a. Let the simple closed curve encircling this arc be $0/1 = 0$. Note that, in choosing $A$, there is a $\mathbb{Z}$-worth of choices up to isotopy; by Lemma 2.2, this ambiguity turns out to be immaterial.

Given a fixed meaning for $1/0$ and $0/1$, as well as an orientation on the 4-punctured projection sphere $S$, every curve on $S$ is determined by a number $p/q \in \mathbb{Q} \cup \{1/0\}$, where $p$ and $q$ are relatively prime. Concretely, this curve can be drawn by marking $q$ ticks on the arcs

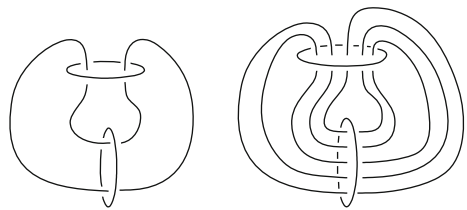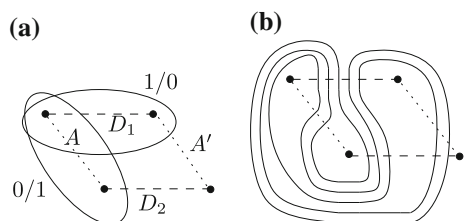**Fig. 3** Examples of links obtained by adding crossing circles to double coil knots and untwisting



**Fig. 4** (a) A framing for the 4-punctured sphere. (b) The curve 2/5

corresponding to $D_1$ and $D_2$, and $p$ ticks on the arcs $A$ and $A'$ of Fig. 4a, and then connecting the dots, as in Fig. 4b.

**Definition 2.1** The three-component link consisting of $C_1$, $C_2$, and the curve of slope $p/q$ will be denoted $L_{p/q}$. Thus, Fig. 3 depicts $L_{1/2}$ and $L_{3/5}$. Note that for $p$, $q$ relatively prime and $0 < p < q$, $L_{p/q}$ is the augmentation of a $(p, q)$ double coil knot. The $(p, q)$ double coil knot with $n_1$ full twists in one generalized twist region and $n_2$ full twists in the other generalized twist region can be recovered from $L_{p/q}$ by performing $1/n_i$ Dehn filling on $C_i$.

**Lemma 2.2** *The link $L_{p/q}$ is isotopic to $L_{k+(p/q)}$, by an isotopy that preserves the projection plane.*

*Proof* In the projection plane, the curves of slope $p/q$ and $k+(p/q)$ are related by performing $k$ half-Dehn twists about the closed curve of slope $1/0$. Note that this curve of slope $1/0$ is the intersection between the projection plane and a 2-sphere $\Sigma$ that separates $D_1$ from $D_2$. Thus, because $\Sigma$ is disjoint from $C_1$ and $C_2$, the Dehn twists about its equator can be realized by an isotopy in $S^3$ that preserves the projection plane and carries $L_{p/q}$ to $L_{k+(p/q)}$.    □

Thus we may assume $0 < p < q$, provided $p/q \notin \{0, \infty\}$. The cases in which $p/q = 0$ or $\infty$ do not lead to hyperbolic links, and so we will assume $q \geq 2$.

**Lemma 2.3** *The augmentations of double coil knots have the following symmetries:*

(a)  $S^3 \backslash L_{p/q}$ *admits an orientation-reversing involution, namely reflection in the projection plane.*
(b)  $S^3 \backslash L_{p/q}$ *admits an orientation-preserving involution interchanging $C_1$ and $C_2$.*
(c)  $S^3 \backslash L_{p/q}$ *is homeomorphic to $S^3 \backslash L_{-p/q}$.*

*Proof* The involution in (a) is immediately visible in Fig. 3. The involution in (b) is a $\pi$-rotation about an axis perpendicular to $S$. Within $S$, the involution is a $\pi$-rotation about two points (in Fig. 4, the center of the parallelogram and the point at infinity), which sends the curve of slope $p/q$ to an isotopic curve. Finally, statement (c) is immediate because $L_{p/q}$ becomes $L_{-p/q}$ when viewed from the other side of the projection plane.    □

2.2 2-Bridge links and augmented 2-bridge links

The links $L_{p/q}$ are related in a fundamental way to 2-bridge links. In order to show this relationship, we present the following method for constructing links.

Let $S$ denote the 4-punctured sphere. Consider $S \times [0, 1]$ embedded in $S^3$, with the framing on $S = S \times \{t\}$ as above, for all $t \in [0, 1]$.

Recall that we may obtain (the complement of) any 2-bridge link by attaching two 2-handles to $S \times [0, 1]$, one along the slope $1/0$ on $S \times \{1\}$, and one along a slope $p/q$ on $S \times \{0\}$. Since Dehn twisting along $1/0$ gives a homeomorphic link, we may assume $p/q \in \mathbb{Q}/\mathbb{Z}$. The continued fraction expansion of $p/q$ now describes an alternating diagram of the 2-bridge link. See [5, Proposition 12.13]. One example is depicted in Fig. 5a.

We modify this construction slightly. Attach a 2-handle to $S \times \{1\}$ along the slope $1/0$ as before. However, on $S \times \{0\}$, chisel out the slope $p/q$. This separates $S \times \{0\}$ into two 2-punctured disks. Glue one 2-punctured disk to the other, gluing the boundary corresponding to the slope $p/q$ to itself, and gluing the other boundary components in pairs. We call this link the *clasped 2-bridge link of slope $p/q$*. See Fig. 5b for an example.

(Note that up to homeomorphism, there are two ways to glue the 2-punctured disks so that the boundary $p/q$ is glued to itself. Either way is acceptable and leads to the same results below: any extra crossing cancels with its mirror image in Proposition 2.5.)
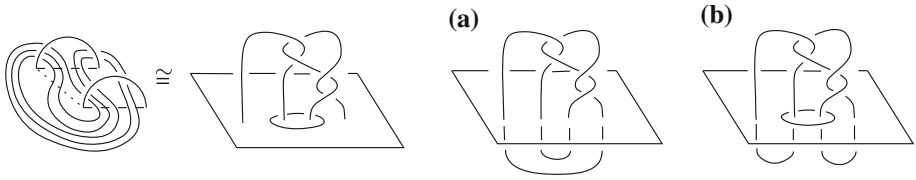
**Fig. 5** (**a**) Constructing the 2-bridge knot of slope 2/5. (**b**) Constructing the clasped 2-bridge link of slope 2/5
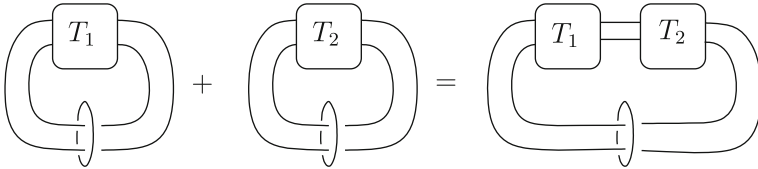


**Fig. 6** A belted sum

*Remark 2.4* Note that the clasped 2-bridge link of slope $p/q$ has a diagram similar to the diagram of the regular 2-bridge link of slope $p/q$, as in Fig. 5. In particular, the diagrams will be identical "above" the embedded surface $S \times \{0\}$, and here we take both diagrams to agree with the standard alternating diagram of the 2-bridge link. On $S \times \{0\}$, the clasped 2-bridge link will have an extra link component, the *clasp* component, which bounds two embedded 2-punctured disks in $S \times \{0\}$. Below $S \times \{0\}$, both diagrams consist of two simple arcs, but they are attached to differing punctures of $S \times \{0\}$ for the 2-bridge link and for the clasped 2-bridge link. Compare the examples in Fig. 5a and b.

Note also that by performing $\pm 1/N$ Dehn filling about the clasp component, we replace the clasp and the two strands it encircles by $N$ full twists of those two strands (in other words, a twist region with $2N$ crossings). By choosing the sign of the Dehn filling appropriately, we can ensure that the result is the alternating diagram of a 2-bridge link of some new slope. Thus, the clasped 2-bridge link of slope $p/q$ can be viewed as an augmented 2-bridge link of some other slope, where we are using the term *augmented* in the sense of Adams [2].

There is a standard way to add two manifolds containing embedded 2-punctured disks, explored by Adams [1]. This is the belted sum. We recall the definition.

Let $M_1$ be the complement of the link in $S^3$ with the following presentation. $T_1$ is some tangle in a 4-punctured sphere. The four punctures of this sphere are connected in a manner as shown on the left in Fig. 6, with a simple closed unknotted curve $B_1$ encircling the two strands. Note $B_1$ bounds a 2-punctured disk $S_1$ in the complement of the link. We will call the link component $B_1$ the *belt component* of the link. We will call the link consisting of $T_1$ and the component $B_1$ a *belted link*. We will only be interested in belted links admitting hyperbolic structures.

Given two hyperbolic belted links with complements $M_1$ and $M_2$, consisting of tangles $T_1$ and $T_2$, belt components $B_1$ and $B_2$, and 2-punctured disks $S_1$ and $S_2$, we form the complement of a new belted link as follows. Cut each manifold $M_1$ and $M_2$ along $S_1$ and $S_2$, respectively. We obtain two manifolds, $\check{M}_1$ and $\check{M}_2$, each with two 2-punctured disks as boundary. There is a unique hyperbolic structure on a 2-punctured disk, hence any two are isometric. This allows us to glue $\check{M}_1$ to $\check{M}_2$ by isometries of their boundaries, gluing $B_1$ to $B_2$. The result is the complement of a new belted link. See Fig. 6. We call this new belted link the *belted sum* of $T_1$ and $T_2$.

**Fig. 7** The belted sum whose
complement is homeomorphic to
$S^3 \backslash L_{p/q}$
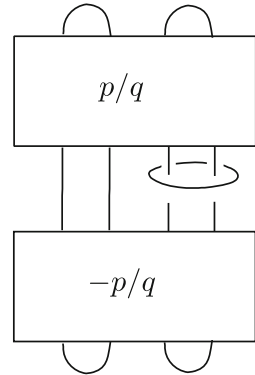


**Proposition 2.5** $S^3 \backslash L_{p/q}$ *is homeomorphic to the belted sum of a clasped 2-bridge link of slope $p/q$ and a clasped 2-bridge link of slope $-p/q$. (See Fig. 7.)*

*Proof* Slice $S^3 \backslash L_{p/q}$ along the projection plane. This cuts the manifold into two pieces, which we call the top half and bottom half, each bounded by a 4-punctured sphere. Let $S$ be a 4-punctured sphere. Note we can embed $S \times [0, 1]$ in the top half in $S^3$ such that $S \times \{0\}$ is embedded on the projection plane, and the punctures of $S \times \{t\}$ correspond to points on the crossing circles $C_1$ and $C_2$.

Alternately, give $S$ the same framing as above, so that $1/0$ corresponds to the curve encircling $D_1$ (or $D_2$), and attach a 2-handle to $S \times \{1\}$ along the slope $1/0$. By our choice of framing, the result is homeomorphic to capping off the halves of arcs $C_1$ and $C_2$. When we chisel out the slope $p/q$ on the projection plane $S \times \{0\}$, the result is a manifold with boundary consisting of two 2-punctured disks. This is homeomorphic to the top half of $S^3 \backslash L_{p/q}$, as in the left of Fig. 5. Thus the top half of $S^3 \backslash L_{p/q}$ sliced along the projection plane is homeomorphic to a clasped 2-bridge link of slope $p/q$ sliced open along the 2-punctured disk bounded by the clasp component.

Similarly, when we consider the bottom half of $S^3 \backslash L_{p/q}$ sliced along the projection plane, we see it is homeomorphic to the clasped 2-bridge link of slope $-p/q$, sliced open along the 2-punctured disk bounded by the clasp component.

Since we glue the 2-punctured disks of the top half to those of the bottom half such that the chiseled-out curve $p/q$ is glued to itself, this is, by definition, a belted sum of the two manifolds.                                                                                                                          □

2.3 Volume bounds for the parent links

Recall that every rational number $p/q \in \mathbb{Q}$ can be expressed as a finite length continued fraction. When $q > p > 0$ and all the terms of the continued fraction are positive, this expression is unique. We define the *length* of the continued fraction to be the number of denominators in this unique continued fraction where all denominators are positive.

**Theorem 2.6** *Let $k$ be the length of the continued fraction expansion of $p/q$, with $0 < p < q$ and $q \geq 2$. Then $L_{p/q}$ is hyperbolic, and*

$$4kv_3 - 1.3536 \leq \text{vol}(S^3 \backslash L_{p/q}) \leq 4kv_8,$$

*where $v_3 = 1.0149\ldots$ is the volume of a regular ideal tetrahedron and $v_8 = 3.6638\ldots$ is the volume of a regular ideal octahedron in $\mathbb{H}^3$.*

*Proof* By Proposition 2.5, $S^3 \backslash L_{p/q}$ is homeomorphic to the belted sum of two clasped 2-bridge links, one of slope $p/q$ and one of slope $-p/q$. By Remark 2.4, a portion of the diagram of each clasped 2-bridge link (essentially, everything away from the clasp) agrees with the alternating diagram of the regular 2-bridge link of slope $p/q$ or $-p/q$, respectively. It is well known that these 2-bridge links will have exactly $k$ twist regions (see, for example Burde and Zieschang [5, Proposition 12.13]). Thus the clasped 2-bridge links will contain $k$ twist regions as well as a separate clasp component. By our restrictions on $q$ and $p$, $k \geq 1$.

Now, again by Remark 2.4, the clasped 2-bridge link can be Dehn filled, along slope $\pm 1/N$ on the clasp component, to give a diagram of a new alternating 2-bridge link $K_N$. The link $K_N$ will have $k + 1$ twist regions, with $2N$ crossings in the new $(k + 1)$-st twist region. As $N$ approaches infinity, the limit in the geometric topology of the $K_N$ will be the original clasped 2-bridge link [27]. Because $k + 1 \geq 2$, each link $K_N$ is hyperbolic (see e.g., [14, Theorem A.1]). Thus its geometric limit is also hyperbolic. Finally, the belted sum of hyperbolic manifolds is hyperbolic. So $L_{p/q}$ is hyperbolic.

Futer and Guéritaud have found bounds on the volumes of 2-bridge knots. By [14, Theorem B.3], the complement of a 2-bridge knot whose standard alternating diagram has $k + 1$ twist regions has volume at least $2(k + 1)v_3 - 2.7066$ and at most $2kv_8$. Since the clasped 2-bridge link of slope $\pm p/q$ is the geometric limit of such manifolds, it satisfies the same volume bounds. Adams observed that the volume of a belted sum of two hyperbolic manifolds is equal to the sum of the volumes of the two pieces [1]. Thus, the volume of $S^3 \backslash L_{p/q}$ is at least $4kv_3 - 1.3536$ and at most $4v_8 k$. $\qquad\qquad\qquad\qquad\square$

2.4 Volume bounds for double coil knots

Let $K$ be a $(p, q)$ double coil knot. Then, by Definition 2.1, $K$ is obtained by $1/n_i$ filling on the component $C_i$ of $L_{p/q}$, $i = 1, 2$. We may bound the volume of $K$ by bounding the change in volume under Dehn filling. Our main tool is the following recent result of the authors, Theorem 1.1 of [11].

**Theorem 2.7** [11] *Let $M$ be a complete, finite-volume hyperbolic manifold with cusps. Suppose $C_1, \ldots, C_k$ are disjoint horoball neighborhoods of some subset of the cusps. Let $s_1, \ldots, s_k$ be slopes on $\partial C_1, \ldots, \partial C_k$, each with length greater than $2\pi$. Denote the minimal slope length by $\ell_{\min}$. Then the manifold $M(s_1, \ldots, s_k)$, obtained by Dehn filling $M$ along $s_1, \ldots, s_k$, is hyperbolic, and*

$$\mathrm{vol}(M(s_1, \ldots, s_k)) \geq \left(1 - \left(\frac{2\pi}{\ell_{\min}}\right)^2\right)^{3/2} \mathrm{vol}(M).$$

We also need the following additional notation. Let $k, n_1, n_2$ be integers. Define

$$n := \min\{|n_1|, |n_2|\} \quad \text{and} \quad \ell := \max\left\{\frac{1}{4} + 4n^2, \ \frac{32\sqrt{2}\,k^2 n^2}{7203}\right\}. \tag{1}$$

Note that the right-hand term of the maximum becomes greater when $k \geq 26$. We may now give volume bounds on double coil knots.

**Theorem 2.8** *Let $K$ be a $(p, q)$ double coil knot, where one generalized twist region contains $n_1$ positive full twists, and the other region contains $n_2$ twists.*

*Let $k$ denote the length of the continued fraction expansion of $p/q$, and let $\ell$ be as in ([1])* *above. Suppose that at least one of the following holds:*

(1)  $|n_i| \geq 4$ *for* $i = 1, 2$.
(2)  $k\,|n_i| \geq 80$ *for* $i = 1, 2$.

*Then $K$ is hyperbolic with volume*

$$\left(1 - \frac{4\pi^2}{\ell}\right)^{3/2} (4kv_3 - 1.3536) \leq \mathrm{vol}(S^3 \backslash K) < 4v_8 k,$$

*where $v_3 = 1.0149\ldots$ is the volume of a regular ideal tetrahedron and $v_8 = 3.6638\ldots$ is* *the volume of a regular ideal octahedron in $\mathbb{H}^3$.*

*Proof* By Definition 2.1, $K$ is obtained by $1/n_i$ filling on the component $C_i$ of $L_{p/q}$. Thus the upper bound on volume follows immediately from Theorem 2.6 and the fact that the volume decreases under Dehn filling [27, Theorem 6.5.6].

For the lower bound, let $\ell_{\min}$ denote the minimum of the lengths of $1/n_1$ and $1/n_2$ in some horoball expansion about the cusps corresponding to $C_1$ and $C_2$. Provided $\ell_{\min} > 2\pi$, Theorem 2.7 implies that $K$ is hyperbolic and:

$$
\begin{aligned}
\mathrm{vol}(S^3 \backslash K) &\geq \left(1 - \left(\frac{2\pi}{\ell_{\min}}\right)^2\right)^{3/2} \mathrm{vol}(S^3 \backslash L_{p/q}) \\
&\geq \left(1 - \left(\frac{2\pi}{\ell_{\min}}\right)^2\right)^{3/2} (4kv_3 - 1.3536).
\end{aligned}
$$

Thus we determine some admissible values of $n_1$, $n_2$, and $k$, for which the slopes $1/n_1$ and $1/n_2$ are both guaranteed to have length at least $2\pi$ under some horoball expansion. First, recall that by Lemma 2.3(a) we may arrange the diagram of $L_{p/q}$ such that the link $L_{p/q}$ is fixed under reflection in the projection plane. It follows immediately from [23, Proposition 3.5] that there exists a horoball expansion about these cusps such that the slope $1/n_i$ has length at least

$$\ell_i \geq \sqrt{1/4 + 4n_i^2}. \tag{2}$$

This quantity is greater than $2\pi$ when $n_i \geq 4$. Hence the conclusion follows in this case.

On the other hand, when $k$ is relatively large, we can get a better estimate on the lengths of the slopes $1/n_1$ and $1/n_2$. In [10, Theorem 4.8], we found bounds on the length of certain arcs on the cusps of 2-bridge knots. In particular, the shortest non-trivial arc running from a meridian back to that meridian in a 2-bridge knot with $(k+1)$ twist regions has length at least $(4\sqrt{6\sqrt{2}})\,k/147$. In other words, the area of the maximal cusp about the knot is at least $(4\sqrt{6\sqrt{2}})\,k \cdot \mu/147$, where $\mu$ is the length of the meridian. Since a clasped 2-bridge link of slope $p/q$ is the geometric limit of 2-bridge knots with $(k+1)$ twist regions, the same estimate applies to the clasped link.

Now, by Proposition 2.5, $L_{p/q}$ is obtained as the belted sum of clasped 2-bridge links of slope $p/q$ and $-p/q$. Consider what happens to the cusps during the gluing process. The cusps about $C_1$ and $C_2$ come from the knot component(s) of the clasped link, i.e. the component(s) which form the 2-bridge link rather than the clasp. The meridians of $C_1$ and $C_2$ agree with meridians of the 2-bridge link, and both have length $\mu$. Furthermore, the total area of the cusps about $C_1$ and $C_2$ is equal to twice the area of the cusp about the 2-bridge knot, namely at least $(8\sqrt{6\sqrt{2}})\,k \cdot \mu/147$. But by Lemma 2.3b, there is a symmetry of $L_{p/q}$

interchanging $C_1$ and $C_2$, hence each of those cusps has area at least $(4\sqrt{6\sqrt{2}})\,k \cdot \mu/147$. As a result, in each of $C_1$ and $C_2$, the shortest non-trivial arc running from a meridian back to that meridian has length at least $(4\sqrt{6\sqrt{2}})\,k/147$.

Finally, note that the slope $1/n_i$ crosses the meridian exactly $|n_i|$ times. Since each non-trivial arc from the meridian to the meridian has length bounded as above, the total length of the slope is at least $(4\sqrt{6\sqrt{2}})\,k|n_i|/147$. When $k|n_i| \geq 80$, the slope $1/n_i$ will be longer than $2\pi$, and the desired volume estimate follows.                                                  □

Theorem 2.9, stated in the introduction, is now an immediate corollary of Theorem 2.8.

**Theorem 2.9** *Let $p$ and $q$ be relatively prime integers with $0 < p < q$, and let $k$ be the length of the continued fraction expansion of $p/q$. Let $K$ be a $(p,q)$ double coil knot, in which each generalized twist region has at least 4 full twists. Then $K$ is hyperbolic, and*

$$0.9718\,k - 0.3241 \;\leq\; \mathrm{vol}(S^3\backslash K) \;<\; 4v_8 k,$$

*Proof* If $|n_i| \geq 4$ for $i = 1, 2$, then $\ell \geq 64.25$ in equation (1). Plugging this estimate into Theorem 2.8, and substituting the numerical values of all the constants in the lower bound on volume, gives the desired result.                                                  □

## 3 Volume and twist number

The twist number of a knot $K$, denoted by $\tau(K)$, is defined to be the minimum twist number over all knot diagrams of $K$; it is clearly an invariant of $K$. In this section, we describe a general construction of hyperbolic knots with bounded volume and arbitrarily large twist number.

**Theorem 3.1** *Let $K$ be a knot in $S^3$, and let $U \subset S^3\backslash K$ be an unknot in $S^3$ with the property that $S^3\backslash(K \cup U)$ is hyperbolic. Suppose that every disk bounded by $U$ intersects $K$ at least three times. Let $K_n$ be the knot obtained by $1/n$ surgery on $U$. Then, for $|n|$ sufficiently large, $K_n$ is a hyperbolic knot, with*

$$\mathrm{vol}(S^3\backslash K_n) < \mathrm{vol}(S^3\backslash(K \cup U)) \quad and \quad \lim_{|n|\to\infty} \tau(K_n) = \infty.$$

*Proof* By Thurston's hyperbolic Dehn surgery theorem [27], $K_n$ is hyperbolic for $|n|$ large enough. Furthermore, because volume decreases strictly under Dehn filling [27, Theorem 6.5.6], we have $\mathrm{vol}(S^3\backslash K_n) < \mathrm{vol}(S^3\backslash(K \cup U))$. Now the complement $S^3\backslash(K \cup U)$ is the geometric limit of $S^3\backslash K_n$, as $|n| \to \infty$. By the proof of the hyperbolic Dehn surgery theorem, for $n$ large enough, the core of the Dehn filling torus (that is, $U$) is the unique minimum-length geodesic in $S^3\backslash K_n$, and the length of this geodesic goes to 0 as $n \to \infty$. Note that, since $K_n$ is obtained from $K$ by twisting about the disk bounded by $U$, this disk intersects $K_n$ the same number of times as $K$, namely at least three.

Now we argue that $\lim_{|n|\to\infty} \tau(K_n) = \infty$. Suppose not: assume that $\tau(K_n)$ is bounded independently of $n$. Then consider the knot $K_n$. Take a diagram of $K_n$ for which the twist number is minimal, equal to $\tau(K_n)$. Encircle each twist region of the diagram by a crossing circle. The result is a *fully augmented link*, and the knot $K_n$ is obtained by Dehn filling this fully augmented link. (For an example with two twist regions, compare Fig. 2, left to Fig. 3, left.)

To obtain the standard diagram of a fully augmented link, we remove all pairs of crossings (full twists) from each crossing circle. (See also, for example, Fig. 6 of [11].) Thus

the standard diagram of a fully augmented link with $\tau$ twist regions consists of $\tau$ crossing circles encircling two strands each, possibly with a single crossing at each crossing circle. This can be represented by a 4-valent graph with $\tau$ vertices, and a choice of crossing at each vertex. Since $\tau(K_n)$ is bounded independently of $n$, there are only finitely many such 4-valent graphs, so only finitely many fully augmented links. As a result, there must be an infinite subsequence of knots $K_n$ that is obtained by surgery on a single augmented link.

Recall that this subsequence converges geometrically to $S^3\backslash(K \cup U)$. Thus, in each twist region in this infinite subsequence, the number of crossings either becomes eventually constant or goes to infinity. Since infinitely many of the $K_n$ are distinct knots, there is at least one twist region whose number of crossings goes to infinity. Furthermore, since the geometric limit of $S^3\backslash K_n$ is the manifold $S^3\backslash(K \cup U)$ with exactly two cusps, there must be exactly one twist region whose number of crossings goes to infinity, for if the number of crossings goes to infinity, the geometric limit yields an additional cusp [27].

Hence we have an infinite subsequence of the knots $K_n$ that is obtained from a 2-component link $K' \cup U'$ by Dehn filling along an unknotted component $U'$. Furthermore, the subsequence $S^3\backslash K_n$ converges geometrically to $S^3\backslash(K' \cup U')$, and the crossing circle $U'$ bounds a disc whose interior is pierced exactly twice by $K'$. Furthermore, under this convergence the core $U'$ eventually becomes the unique minimal geodesic in $S^3\backslash K_n$.

Now recall that for $|n|$ sufficiently large, $U$ is also the unique minimal-length geodesic in $S^3\backslash K_n$. We conclude that for $|n|$ large enough there must be an isometry $S^3\backslash K_n \longrightarrow S^3\backslash K_n$ that maps $U$ onto $U'$. However, $U'$ bounds a disk whose interior is punctured twice by $K_n$, whereas $U$ does not. This is a contradiction.                                                        $\square$

One way to construct a sequence of knots satisfying Theorem 3.1 is the following.

**Corollary 3.2** *Fix an integer $m \geq 3$. Then there is a sequence $K_n$ of hyperbolic closed $m$-braids, such that* $\mathrm{vol}(S^3\backslash K_n)$ *is bounded but* $\tau(K_n)$ *is unbounded.*

*Proof* Given a closed $m$-braid $K$, let $A$ be the braid axis of $K$. That is: $S^3\backslash A$ is a solid torus swept out by meridian disks, with each disk intersecting $K$ in $m$ points. The complement $S^3\backslash(K \cup A)$ is a fiber bundle over $S^1$, with fiber an $m$-punctured disk. By a theorem of Thurston [26], this manifold will be hyperbolic whenever the monodromy is pseudo-Anosov. Furthermore, since the fiber minimizes the Thurston norm within its homology class, the unknot $A$ does not bound a disk meeting $K$ in fewer than $m$ points. Thus, Theorem 3.1 applies, and the sequence of knots $K_n$ obtained by $1/n$ filling on $A$ has bounded volume but unbounded twist number.                                                        $\square$

For the double coil knots studied in the last section, Theorem 3.1 applies to give:

**Theorem 3.3** *The volumes of hyperbolic double coil knots are not effectively predicted by either the twist number or generalized twist number. More precisely:*

(a)   *For any $q \geq 3$, and any $p$ relatively prime to $q$, there exists a sequence $K_n$ of $(p, q)$ double coil knots such that $\tau(K_n) \to \infty$ while $\mathrm{vol}(S^3\backslash K_n)$ stays bounded.*
(b)   *All double coils have generalized twist number 2, but their volumes are unbounded.*

*Proof* Statement (b) is an immediate consequence of Definition 1.3 and Theorem 2.9.

For statement (a), consider the sequence $K_n$ of double coil knots obtained from $L_{p/q}$ by $1/n$ filling on the circle $C_1$ and $1/6$ filling on $C_2$. When $n \geq 4$, Theorem 2.9 implies that each $K_n$ is hyperbolic. The volumes of $S^3\backslash K_n$ are bounded above by the volume of $S^3\backslash L_{p/q}$. To apply Theorem 3.1 and show that the twist number of $K_n$ is unbounded, we need to show that every disk bounded by $C_1$ meets $K_n$ at least three times.

Suppose, for a contradiction, that $D$ is a disk in $S^3$ whose boundary is $C_1$, and such that $|K_n \cap D| \leq 2$. Since $S^3 \backslash K_n$ is hyperbolic, it cannot contain any essential disks or annuli. Thus $K_n$ meets $D$ exactly twice. We assume that $D$ has been moved by isotopy into a position that minimizes its intersection number with $C_2$, and consider two cases.

*Case 1* $D$ is disjoint from $C_2$.

Then, when $C_1$ and $C_2$ are drilled out of $S^3 \backslash K_n$, $D$ becomes a disk in $S^3 \backslash L_{p/q}$ that intersects $K$ twice, where $K$ is the planar curve of slope $p/q$ that will become $K_n$ after Dehn filling on $C_1$ and $C_2$. Consider the standard diagram of $L_{p/q}$, with all full-twists removed from generalized twist regions. Isotope $D$ so that it intersects the projection plane of this diagram transversely a minimal number of times. Then the intersection between $D$ and the 4-punctured projection sphere $S$ consists of some number of simple closed curves, as well as exactly one arc $\alpha$ connecting two of the punctures of $S$. (These punctures are the intersections between $C_1$ and the projection plane—see Fig. 4.)

Since $D$ intersects $K$ twice, and $K$ lies in the projection plane as a curve of slope $p/q$, this arc $\alpha \subset D$ must intersect the curve of slope $p/q$ at most twice. On the other hand, since $\alpha$ lies in a disk whose boundary is $C_1$, $\alpha$ must be isotopic to one half of $C_1$, in other words to an arc of slope $1/0$ in $S$. But it is well-known (for example, by passing to the universal abelian cover $\mathbb{R}^2 \backslash \mathbb{Z}^2$) that in a 4-punctured sphere, the arc of slope $1/0$ and the closed curve of slope $p/q$ must intersect at least $q$ times. Since $q \geq 3$, this is a contradiction.

*Case 2* $D$ is not disjoint from $C_2$.

Let $E = D \cap (S^3 \backslash L_{p/q})$. Then $E$ is a sphere with $(r + 3)$ holes, where one boundary circle is at the cusp of $C_1$, two boundary components are at the cusp of $K$, and $r$ boundary components are at the cusp of $C_2$. Consider the length $\ell$ of the Dehn filling slope along $C_2$, where $r \geq 1$ boundary circles of $E$ run in parallel along the cusp. A result of Agol and Lackenby (see [3, Theorem 5.1] or [19, Lemma 3.3]) implies that the total length of those circles is

$$r \ell \leq -6 \chi(E) = 6(r + 1) \leq 12r.$$

Thus $\ell \leq 12$. On the other hand, since we are filling $C_2$ along slope $1/6$, Eq. (2) above implies that

$$\ell \geq \sqrt{1/4 + 4 \cdot 6^2} = \sqrt{144.25} > 12.$$

Therefore, in this case as well as in Case 1, we obtain a contradiction.    □

## 4 Spectral geometry

In this section, we investigate the spectral geometry of double coil knot complements. Recall that, for a Riemannian manifold $M$, $\lambda_1(M)$ is defined to be the smallest positive eigenvalue of the Laplace–Beltrami operator $\Delta f = -\text{div grad} f$. When $M$ is a hyperbolic 3-manifold, it is known that $\lambda_1(M)$ has many connections to the volume of $M$. The following result is essentially a combination of theorems by Schoen [25], Dodziuk and Randol [9], Lackenby [21], and Buser [6].

**Theorem 4.1** *Let M be an oriented, finite-volume hyperbolic* 3-*manifold. Then*

$$\frac{\pi^2/2^{50}}{\text{vol}(M)^2} \ \le \ \lambda_1(M) \ \le \ 32\pi \, \frac{g(M)-1}{\text{vol}(M)} + 640\pi^2 \, \frac{(g(M)-1)^2}{\text{vol}(M)^2},$$

*where* $g(M)$ *is the Heegaard genus of* $M$.

To write down the proof of Theorem 4.1, we need the following fact.

**Lemma 4.2** *An oriented, finite-volume hyperbolic* 3-*manifold M satisfies* $\text{vol}(M) > \pi/2^{25}$.

*Proof* Gabai, Meyerhoff, and Milley recently showed [13] that the unique lowest-volume orientable hyperbolic 3-manifold is the Weeks manifold of volume $\approx 0.9427$. This is the culmination of many increasingly sharp estimates, by a number of hyperbolic geometers. In fact, Meyerhoff's 1984 result [22] that $\text{vol}(M) \ge 0.00064$ is several orders of magnitude larger than necessary for this lemma.                                                        □

*Proof* (Proof of Theorem 4.1) Dodziuk and Randol [9] showed that for all finite-volume, hyperbolic $n$-manifolds (where $n \ge 3$), $\lambda_1(M) \ge A(n)/\text{vol}(M)^2$, where the constant $A(n)$ depends only on the dimension $n$. To estimate $A(3)$ for dimension 3, we rely on the work of Schoen, who gave an explicit estimate for $\lambda_1(M)$ when $M$ is closed and negatively curved [25]. In the special case where $M$ is a closed, hyperbolic 3-manifold, his theorem says that

$$\lambda_1(M) \ \ge \ \min \left\{ 1, \ \frac{\pi^2}{2^{50}} \cdot \frac{1}{\text{vol}(M)^2} \right\} \ \ge \ \frac{\pi^2/2^{50}}{\text{vol}(M)^2},$$

where the second inequality is Lemma 4.2. This completes the proof of the lower bound on $\lambda_1(M)$ in the case where $M$ is closed.

Now, suppose that $M$ has cusps. We may assume that $\lambda_1(M) < 1$; otherwise, $\lambda_1(M)$ already satisfies the desired lower bound by Lemma 4.2. Let $N_i$ be a sequence of closed manifolds obtained by Dehn filling $M$, along slopes whose lengths tend to infinity. Thurston's Dehn surgery theorem [27] implies that the manifolds $N_i$ approach $M$ in the geometric topology; in particular, $\text{vol}(N_i) \to \text{vol}(M)$. Meanwhile, assuming that $\lambda_1(M) < 1$, Colbois and Courtois [8, Theorem 3.1] showed that $\lambda_1(N_i) \to \lambda_1(M)$. Thus, since the lower bound on $\lambda_1$ holds for each closed $N_i$, it also holds for $M$.

The upper bound on $\lambda_1(M)$ is a combination of results by Buser [6] and Lackenby [21]. Buser proved an inequality relating $\lambda_1(M)$ to the *Cheeger constant* $h(M)$, defined by

$$h(M) := \inf \left\{ \frac{\text{area}(S)}{\min(V_1, V_2)} \right\},$$

where $S$ is a separating surface in $M$, and $V_1$, $V_2$ are the volumes of the two pieces separated by $S$. Buser's result [6] says that

$$\lambda_1(M) \le 4h(M) + 10h(M)^2.$$

More recently, Lackenby showed [21, Theorem 4.1] that if a hyperbolic manifold $M$ has a genus-$g$ Heegaard splitting,

$$h(M) \le \frac{8\pi(g-1)}{\text{vol}(M)}.$$

Plugging this estimate into Buser's inequality yields the upper bound on $\lambda_1(M)$.                □

For double coil knots, Theorem 4.1 implies the following result.

**Theorem 4.3** *Let K be a hyperbolic double coil knot. Then*

$$\frac{A_1}{\text{vol}(S^3\backslash K)^2} \ \leq \ \lambda_1(S^3\backslash K) \ \leq \ \frac{A_2}{\text{vol}(S^3\backslash K)},$$

*where $A_1 \geq 8.76 \times 10^{-15}$ and $A_2 \leq 12650$.*

*Proof* The lower bound on $\lambda_1$ is a restatement of Theorem 4.1. Note $\pi^2/2^{50} \approx 8.765 \times 10^{-15}$.

To establish the upper bound on $\lambda_1$, we bound the Heegaard genus of $S^3\backslash K$. Recall that $K$ is obtained by Dehn filling two components of the link $L_{p/q}$ depicted in Fig. 7. Since each of the boxes in Fig. 7 contains a braid, the figure is a 3-bridge diagram of $L_{p/q}$. It is well-known that a $g$-bridge link $L$ has Heegaard genus at most $g$. (One standard way to obtain a Heegaard surface is to connect the maxima in a $g$-bridge diagram of $L$ by $g - 1$ arcs, thicken the union of $L$ and these arcs, and take the boundary of the resulting genus-$g$ handlebody. The exterior of this handlebody is unknotted, because $L$ was in bridge position. See [24, Figure 1].) Thus, $S^3\backslash L_{p/q}$ has Heegaard genus at most 3. Since Heegaard genus can only go down under Dehn filling, $S^3\backslash K$ also has Heegaard genus at most 3.

Plugging $g(S^3\backslash K) \leq 3$ into Theorem 4.1, we obtain

$$\begin{aligned}
\lambda_1(S^3\backslash K) &\leq \frac{64\pi}{\text{vol}(S^3\backslash K)} + \frac{2560\pi^2}{\text{vol}(S^3\backslash K)^2}\\
&\leq \frac{64\pi}{\text{vol}(S^3\backslash K)} + \frac{2560\pi^2}{\text{vol}(S^3\backslash K)\cdot 2v_3}\\
&< \frac{12650}{\text{vol}(S^3\backslash K)},
\end{aligned}$$

where the second inequality follows because the smallest-volume knot is the figure-8 knot, with $\text{vol}(S^3\backslash K) = 2v_3$ [7]. □

A collection $\{M_i\}$ of hyperbolic 3-manifolds is called an *expanding family* if $\inf\{\lambda_1(M_i)\} > 0$, that is, $\lambda_1(M_i)$ is bounded away from 0. With this notation, Theorem 4.3 has the following immediate corollary.

**Corollary 4.4** *Let $\{K_i\}$ be a collection of hyperbolic double coil knots. Then $\{\lambda_1(S^3\backslash K_i)\}$ is bounded away from 0 if and only if $\{\text{vol}(S^3\backslash K_i)\}$ is bounded above. In other words, the knots $\{K_i\}$ form an expanding family if and only if their volumes are bounded.*

Corollary 4.4 is significant in light of recent work of Lackenby [18]. He showed that for two large families of hyperbolic links (namely, alternating links and highly twisted links), $\lambda_1(S^3\backslash K)$ is bounded above in terms of the inverse of the twist number of a sufficiently reduced diagram. Because the volumes of these links are also governed by the twist number [11,20], it follows that alternating and highly twisted links form an expanding family if and only if their volumes are bounded [18, Corollary 1.7]. Corollary 4.4 is the analogous result for double coil knots.

On the other hand, by Theorem 3.3, the volumes of double coil knots are not governed by the twist number in any meaningful sense. Thus, combining Theorem 3.3 with Corollary 4.4 yields the following result.

**Corollary 4.5** *The spectrum of the Laplacian of hyperbolic double coil knots is not effectively predicted by either the twist number or generalized twist number. More precisely:*

(a)   *For any $q \geq 3$, and any $p$ relatively prime to $q$, there exists a sequence $K_n$ of $(p, q)$ double coils such that $\tau(K_n) \to \infty$ while $\lambda_1(S^3\backslash K_n)$ is bounded away from 0 and $\infty$.*

(b)  *All double coil knots have generalized twist number* 2, *but the infimum of* $\{\lambda_1(S^3 \setminus K_n)\}$
     *is zero.*

*Proof*  Each part of this corollary follows by combining Theorem 4.3 with the corresponding
part of Theorem 3.3.                                                                    □

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