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CUBULATING RANDOM QUOTIENTS OF HYPERBOLIC CUBULATED GROUPS

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Abstract. We show that low-density random quotients of cubulated hyperbolic groups are again cubulated (and hyperbolic). Ingredients of the proof include cubical small-cancellation theory, the exponential growth of conjugacy classes, and the statement that hyperplane stabilizers grow exponentially more slowly than the ambient cubical group.

1. Introduction

Gromov introduced the *density model* of random groups [Gro93, Chapter 9]. Given a free group F, let $S_\ell(F)$ be the set of length ℓ words. For a density $d \in (0,1)$, choose a set $R \subset S_\ell$ by selecting $\lfloor |S_\ell|^d \rfloor$ elements from S_ℓ, uniformly and independently. The associated *random group at density* d is the quotient $F/\langle\langle R \rangle\rangle$. Thus, for a rank r free group, a density d random quotient has the form $\langle a_1, \ldots, a_r \mid g_1, \ldots, g_k \rangle$, where $k \sim (2r-1)^{d\ell}$ and the g_i are independently chosen random words.

Gromov proved that with overwhelming probability as $\ell \to \infty$, random groups at density d are hyperbolic when $d < \frac{1}{2}$, and trivial or $\mathbb{Z}_2$ when $d > \frac{1}{2}$. Ollivier proved that the same phase transition at density $\frac{1}{2}$ occurs in quotients of any torsion-free hyperbolic group [Oll04]. While Gromov's intention was perhaps to illustrate the ubiquity of hyperbolic groups, his construction initiated a fertile topic of study. For instance, Żuk showed that random groups at density $d > \frac{1}{3}$ satisfy Kazhdan's property (T), with overwhelming probability [Żuk03]. See Kotowski and Kotowski [KK13] for full details, and Ollivier [Oll05] for an excellent survey of related topics. Very recently, Ashcroft extended this result, proving that random quotients of a hyperbolic group at density $d > \frac{1}{3}$ have property (T) [Ash22a].

The property of being *cocompactly cubulated*—acting properly and cocompactly on a CAT(0) cube complex—can be viewed as a strong negation of property (T) [NR98]. In this direction, a simple computation shows that with overwhelming probability as $\ell \to \infty$, random groups at density $d < \frac{1}{12}$ satisfy the $C'(\frac{1}{6})$ small-cancellation condition, hence are cocompactly cubulated by [Wis04]. Ollivier and Wise [OW11] showed that the same conclusion holds at density $d < \frac{1}{6}$. A series of papers by Mackay and Przytycki [MP15], Montee [Mon23], and Ashcroft [Ash22b] produced a nontrivial action on a CAT(0) cube complex at density $d < \frac{1}{4}$. See also Odrzygóźdź [Odr18] and Duong [Duo17] for cubulation results in the square probability model.

The purpose of this text is to extend these cubulation results from quotients of a free group (the fundamental group of a graph) to quotients of a cubulated

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hyperbolic group. Interestingly, our density statement is entangled with the relationship between the growth of a hyperbolic group G and the growth of hyperplane stabilizers in G.

1.1. **Main result.** Our main theorem is the following.

Theorem 1.1. *Let $G = \pi_1 X$, where X is a compact nonpositively curved cube complex, and suppose that G is hyperbolic. Let b be the growth exponent of G with respect to the universal cover $\widetilde{X}$. Let a be the maximal growth exponent of a stabilizer of an essential hyperplane of $\widetilde{X}$. Let $k \leq e^{c\ell}$, where*

$$c \,<\, \min\left\{\frac{(b-a)}{20}, \frac{b}{41}\right\}.$$

Then with overwhelming probability as $\ell \to \infty$, for any set of conjugacy classes $[g_1], \dots, [g_k]$ with translation length $|g_i| \leq \ell$, the group $G/\langle\langle g_1, \dots, g_k\rangle\rangle$ is hyperbolic and is the fundamental group of a compact, nonpositively curved cube complex.

In Theorem 1.1, $|g_i|$ is the translation length of g_i acting on the universal cover $\widetilde{X}$; see Definition 2.1. The growth exponent of G with respect to $\widetilde{X}$ is a constant $b > 0$ such that the number of G-orbit points in an ℓ-ball in $\widetilde{X}$ is approximately $e^{b\ell}$. See Definition 2.2 and Theorem 2.4 for a more precise characterization. The growth exponent of a hyperplane stabilizer is defined analogously.

A hyperplane $\widetilde{U} \subset \widetilde{X}$ is *essential* if $\widetilde{X}$ is not contained in a finite neighborhood of $\widetilde{U}$. A typical inessential hyperplane arises when $\widetilde{X} \cong \widetilde{Y} \times \widetilde{F}$, where $\widetilde{F}$ is finite and nontrivial. After subdividing, a compact nonpositively curved cube complex X always deformation retracts to X', where all hyperplanes of X' are essential.

The model of randomness employed in Theorem 1.1 is that we are sampling uniformly from the set of conjugacy classes whose translation length on $\widetilde{X}$ is at most ℓ. This departs from Gromov's density model in two small ways: we are permitting relators whose translation length is less than ℓ, and we are only counting one relator per conjugacy class. Ultimately, the exponential growth rate of balls in $\widetilde{X}$ is the same as the growth rate of spheres, and the growth of group elements is nearly the same as the growth of conjugacy classes (Theorem 2.4). Therefore, small variations in the model of randomness tend to have no effect on the conclusions one can reach about random quotients. See Ollivier ([Oll04, Sections 4 and 5] and [Oll05, Section I.2.c]) for a detailed and axiomatic discussion of this phenomenon.

The conclusion of Theorem 1.1—that $\overline{G} = G/\langle\langle g_1, \dots, g_k\rangle\rangle$ is hyperbolic and cocompactly cubulated—has powerful consequences. A theorem of Agol [Ago13] implies $\overline{G}$ is *virtually special*, meaning $\overline{G}$ virtually embeds into a right-angled Artin group. Then, by a theorem of Haglund and Wise [HW10], all quasiconvex subgroups of $\overline{G}$ are separable. Furthermore, $\overline{G}$ is linear over $\mathbb{Z}$ [HW99, DJ00] and virtually surjects the free group F_2 [AM15].

1.2. **Density and optimality of constants.** In Theorem 1.1, the density of the random presentation $G/\langle\langle g_1, \dots, g_k\rangle\rangle$ is $\frac{c}{b}$. Thus all densities in the conclusion of the theorem are at most $\frac{1}{41}$, and might be lower, depending on the value of a. This is considerably lower than the densities appearing in the theorems surveyed at the start of Section 1. The primary reason for needing the low density is that Theorem 1.1 is proved by establishing that random quotients of G satisfy the $C'(\frac{1}{20})$ cubical small cancellation condition. (See Definition 3.2 for the definition and

Theorem 3.5 for the precise statement that $C'(\frac{1}{20})$ plus several mild hypotheses implies cubulation of the quotient.) Indeed, the probabilistic pigeonhole principle [Oll05, p. 31] implies that at any density larger than $\frac{1}{40}$ there will almost surely be pieces that fellow-travel for more than $\frac{1}{20}$ of their length. Thus our density hypotheses are nearly optimal for ensuring $C'(\frac{1}{20})$ small cancellation.

Strengthening Theorem 1.1 beyond density $\frac{1}{40}$ would require one of two improvements. First, one could attempt to strengthen Theorem 3.5 and establish the cubulation of $C'(\alpha)$ quotients for some parameter $\alpha > \frac{1}{20}$. Second, one could move away from small cancellation theory entirely, for instance by employing isoperimetric inequalities for van Kampen diagrams as in the work of Ollivier and Wise [OW11]. This is also the approach employed in Ollivier's work on quotients of torsion-free hyperbolic groups [Oll04, Thm 3], which applies at all densities up to $\frac{1}{2}$ but only ensures the hyperbolicity of the quotient.

We remark that so long as $G \ncong \mathbb{Z}$, the main theorem is nonvacuous, meaning $a < b$, because hyperplane stabilizers in G have strictly lower growth exponents than G itself. See [DFW19, Thm 1.3] and Theorem 2.3. Thus we may always pick $c > 0$ in Theorem 1.1. This leads to Corollary 1.2.

Corollary 1.2. *Let X be a compact nonpositively curved cube complex, such that $\pi_1 X$ is hyperbolic and nonelementary. Then, at a sufficiently low density, generic quotients of $\pi_1 X$ are cocompactly cubulated and hyperbolic.*

For instance, if G is a surface group and the hyperplane stabilizers are cyclic (which occurs in the standard cubulations of G), we have $a = 0$, hence random quotients of G are cubulated and hyperbolic at density $\frac{1}{41}$. This is far lower than the density of $\frac{1}{6}$ at which the corresponding conclusion is known for quotients of free groups. This supports our belief that the constants in Theorem 1.1 can be improved considerably, especially for surface groups. See Problems 7.1 and 7.2.

1.3. **Actions on other metric spaces.** The growth of a group G is highly sensitive to the choice of metric space on which G acts. For instance, hyperbolic manifold groups in dimension $n \geq 3$ admit canonical geometric actions on a hyperbolic space $\Upsilon = \mathbb{H}^n$, but any cubulated group admits infinitely many distinct actions on nonisometric cube complexes. We may wish to study quotients of G by some number of relators that are sampled with respect to length in Υ rather than a cube complex $\widetilde{X}$. Although lengths in $\widetilde{X}$ and Υ can be compared via an (equivariant) quasiisometry, measuring lengths in the two spaces can lead to rather distinct samples of short words.

Nonetheless, there is an analogue of Theorem 1.1 for sampling words in G with respect to an action on Υ. To formulate Theorem 1.3, we introduce the following nonstandard quantification of quasiisometries. A *λ–quasiisometry* is a coarsely surjective function $f\colon \widetilde{X} \to \Upsilon$ such that there exist positive constants $\lambda_1, \lambda_2, \epsilon$ with $\lambda_1\lambda_2 = \lambda$, where every pair of points $x, y \in X$ satisfies

$$\frac{1}{\lambda_1}\mathsf{d}_{\widetilde{X}}(x,y) - \epsilon \;\leq\; \mathsf{d}_{\Upsilon}(f(x), f(y)) \;\leq\; \lambda_2 \mathsf{d}_{\widetilde{X}}(x,y) + \epsilon. \tag{1.1}$$

The product $\lambda_1\lambda_2 = \lambda$ remains unchanged if the metric on one of the spaces $\widetilde{X}$ or Υ is rescaled by a multiplicative constant. More generally, $\lambda \geq 1$ has the following meaning. If G acts properly and cocompactly on both $\widetilde{X}$ and Υ, then these actions induce pseudometrics d_1, d_2 on G itself. We call these pseudometrics *roughly similar* if they are related by a G-equivariant 1–quasiisometry. The space of rough similarity classes of pseudometrics on G is itself an interesting metric space $\mathscr{D}(G)$, studied topologically since the work of Furman [Fur02], and metrically since the work of Reyes [Rey23, Def 1.2]. In Reyes's natural metric on $\mathscr{D}(G)$, the distance between $[d_1]$ and $[d_2]$ is precisely $\log \lambda$ for the optimal constant $\lambda = \lambda_1\lambda_2$ in a G-equivariant quasiisometry $\widetilde{X} \to \Upsilon$. See [Rey23, CR23a, BR].

We are interested in quotients of G, where the conjugacy classes of relators are drawn uniformly from among all elements of Υ-length less than ℓ. At sufficiently low density, depending on λ, these quotients are again hyperbolic and cubulated.

Theorem 1.3. *Let $G = \pi_1 X$, where X is a compact nonpositively curved cube complex, and suppose that G is hyperbolic. Suppose that G also acts properly and cocompactly on a geodesic metric space Υ, where every nontrivial element of G stabilizes a geodesic axis. Suppose that there is a G-equivariant λ–quasiisometry $\widetilde{X} \to \Upsilon$.*

Let b be the growth exponent of G with respect to Υ, and let a be the maximal growth exponent in Υ of a stabilizer of an essential hyperplane of $\widetilde{X}$. Let $k \le e^{c\ell}$, where

$$c < \min\left\{\frac{(b-a)}{20\lambda}, \frac{b}{40\lambda+1}\right\}.$$

Then with overwhelming probability as $\ell \to \infty$, for any set of conjugacy classes $[g_1], \dots, [g_k]$ with translation length $|g_i|_\Upsilon \le \ell$, the group $\overline{G} = G/\langle\langle g_1, \dots, g_k\rangle\rangle$ is hyperbolic and is the fundamental group of a compact, nonpositively curved cube complex.

As with Theorem 1.1, this result is nonvacuous, because hyperplane stabilizers in G have strictly lower exponential growth rates (with respect to Υ) than G itself. By Theorem 2.3, we have $b > a$, hence we may choose $c > 0$.

As a special case, suppose that a group G preserves a tiling of $\mathbb{H}^2$ by regular right-angled pentagons. Let $\widetilde{X}$ be the square complex dual to the pentagonal tiling. Then every hyperplane is a line, hence hyperplane stabilizers are cyclic and have exponential growth rate $a = 0$. By a theorem of Huber [Hub59], generalized by Margulis [Mar69], the growth exponent of G with respect to $\mathbb{H}^2$ is $b = 1$. Furthermore, in Proposition 6.1, we show that the optimal multiplicative constant in a λ–quasiisometry from $\widetilde{X}$ to $\mathbb{H}^2$ is $\lambda \approx 1.5627$. Consequently, Theorem 1.3 has the following corollary.

Corollary 1.4. *Let S be a hyperbolic surface tiled by regular right-angled pentagons, and let $G = \pi_1 S$. For a number $\ell \gg 0$, let $k \le e^{\ell/63.51}$, and let $[g_1], \dots, [g_k]$ be conjugacy classes in G, chosen uniformly at random from among those of $\widetilde{S}$-length at most ℓ. Then with overwhelming probability as $\ell \to \infty$, the group $\overline{G} = G/\langle\langle g_1, \dots, g_k\rangle\rangle$ is hyperbolic and is the fundamental group of a compact, nonpositively curved cube complex.*

In particular, Corollary 1.4 says that quotients of surface groups, with words sampled from a pentagonal hyperbolic metric, are cubulated and hyperbolic at density $\frac{1}{64}$. This is somewhat worse than sampling with respect to a cubical metric, where we obtain the same conclusion at density $\frac{1}{41}$; see the discussion after Corollary 1.2. The multiplicative gap between these densities is essentially the constant λ.

Studying the effect of quasiisometry constants on density statements about quotients of G has led us to conjecture that there exist cubulations of a hyperbolic manifold group with quasiisometry constant λ arbitrarily close to 1. Equivalently, the points of $\mathscr{D}(G)$ corresponding to cocompact hyperbolic structures are limit points of metrics coming from cocompact cubulations. See Conjecture 7.8 and the subsequent discussion. This conjecture has been recently proved by Brody and Reyes [BR].

1.4. **Overview.** Section 2 reviews several results about the growth of a hyperbolic group, including the existence of a growth exponent and the statement that infinite-index quasiconvex subgroups have a lower growth exponent than the ambient group. Section 3 reviews the definitions of cubical small-cancellation theory and proves Theorem 3.5, a nonprobabilistic cubulation criterion for $C'(\frac{1}{20})$ small-cancellation quotients of a cubulated group. This criterion is of independent interest, and has already been used in the work of Jankiewicz and Wise [JW22].

The probabilistic arguments supporting the proof of Theorem 1.1 are contained in Section 4. In that section, we control the sizes of pieces in a generic cubical presentation and show that below a certain density, a cubical presentation is $C'(\frac{1}{20})$, hence the quotient group is hyperbolic and cubulated. All of the geometric arguments in that section are coarse in nature, and it becomes natural to work with a certain generalization of pieces called *loose pieces*. The study of loose pieces also permits a translation between a G-action on a cube complex $\widetilde{X}$ and a G-action on a more general metric space Υ. We undertake this translation in Section 5, where we prove Theorem 1.3. Finally, in Section 6, we find the optimal multiplicative constants in a quasiisometry between a pentagonal tiling of $\mathbb{H}^2$ and the dual cube complex, proving Proposition 6.1 and Corollary 1.4.

In Section 7, we collect some problems and questions motivated by these results.

2. Growth

This section collects several definitions and results about the growth of hyperbolic groups. None of the results recalled here are original.

Definition 2.1 (Translation lengths)**.** Let G be a group acting properly and cocompactly on a metric space Υ, and let $g \in G$ be an infinite-order element. The *translation length* of g is defined to be $|g|_\Upsilon = \inf\{\mathsf{d}(x, gx) : x \in \Upsilon\}$. The *stable translation length* of g is

$$[\![g]\!]_\Upsilon = \lim_{n\to\infty} \frac{\mathsf{d}(x, g^n x)}{n},$$

for an arbitrary $x \in \Upsilon$. It is a standard property of isometries of metric spaces that the limit exists and is independent of x [BH99, page 230].

Observe that each of $|g|_\Upsilon$ and $[\![g]\!]_\Upsilon$ only depends on the conjugacy class $[g]$. By triangle inequalities, $[\![g]\!]_\Upsilon \le |g|_\Upsilon$ for every g. In addition, when Υ is hyperbolic, there is a constant C such that the following holds for every infinite-order $g \in G$:

$$[\![g]\!]_\Upsilon \ \le\ |g|_\Upsilon \ \le\ [\![g]\!]_\Upsilon + C.$$

Finally, if g stabilizes a geodesic axis in Υ (as will typically be the case in our applications), we have $[\![g]\!]_\Upsilon = |g|_\Upsilon$.

Definition 2.2 (Growth)**.** Let G be a finitely generated group acting properly and cocompactly on a metric space Υ. Fix a basepoint $x \in \Upsilon$ and a subset $H \subset G$. The *growth function* of H with respect to Υ is the function $f_{H,\Upsilon} : \mathbb{N} \to \mathbb{N}$ defined by

$$f_{H,\Upsilon}(n) = \#\left\{h \in H : \mathsf{d}_\Upsilon(x,\, hx) \le n\right\}.$$

Since G is a quotient of a finite-rank free group, and the action on Υ is proper, the growth function $f_{H,\Upsilon}$ is no larger than exponential. Thus it makes sense to consider the logarithm of f. The *growth exponent* of H with respect to Υ is

$$\xi_H(\Upsilon) = \lim_{n\to\infty} \frac{\log f_{H,\Upsilon}(n)}{n},$$

whenever the limit exists. We emphasize that the limit depends a great deal on Υ. However, triangle inequalities in Υ imply that $\xi_H(\Upsilon)$ is independent of the basepoint.

Many results in the literature are expressed in terms of the *growth rate* $\lambda_H(\Upsilon) = \lim \sqrt[n]{f_{H,\Upsilon}(n)} = e^{\xi_H(\Upsilon)}$ instead of the growth exponent $\xi_H(\Upsilon)$. However, the two notions carry the same information.

The following result was proved by Dahmani and the authors [DFW19, Thm 1.1], and independently by Matsuzaki, Yabuki, and Jaerisch [MYJ20, Cor 2.8]. See also [DFW19, Thm 1.3] for a statement that does not assume G is hyperbolic, but does assume Υ is a CAT(0) cube complex and H is a hyperplane stabilizer.

Theorem 2.3. *Let G be a nonelementary hyperbolic group acting properly and cocompactly on a metric space Υ. Let H be a quasiconvex subgroup of infinite index. Then the growth exponents ξ_H and ξ_G exist, and*

$$\xi_H(\Upsilon) < \xi_G(\Upsilon).$$

Theorem 2.4 combines two results of Coornaert and Knieper [Coo93, CK02]. Recall that a nontrivial element $g \in G$ is *primitive* if $g \neq h^n$ for any $n > 1$. Primitive conjugacy classes are defined similarly.

Theorem 2.4. *Let G be a nonelementary group acting properly and cocompactly on a δ-hyperbolic metric space Υ. Then, there exist positive constants A, B, b, n_0, where $b = \xi_G(\Upsilon)$, such that the following hold for $n \geq n_0$.*

(1) *The total number $f_{G,\Upsilon}(n)$ of elements that translate a basepoint $x \in \Upsilon$ by distance at most n satisfies*

$$Ae^{bn} \le f_{G,\Upsilon}(n) \le Be^{bn}.$$

(2) *The number p_n of primitive conjugacy classes of translation length at most n satisfies*

$$A\frac{e^{bn}}{n} \le p_n \le Be^{bn}.$$

Proof. Conclusion (1) is due to Coornaert [Coo93, Thm 7.12]. Conclusion (2) is due to Coornaert and Knieper [CK02, Thm 1.1]. ☐

Remark 2.5. One consequence of Theorem 2.4 (1) is that when Υ is a cell complex, we get the same upper and lower bounds (with a modified upper constant B) on the number of vertices in a metric ball in Υ, where the vertices being counted are not required to be in the G-orbit of the basepoint x. This holds because of the cocompactness of the G-action.

Remark 2.6. The upper bound on the number p_n of primitive classes in Theorem 2.4 (2) implies that the number of *nonprimitive* conjugacy classes of translation length at most n is bounded above by

$$\sum_{j=1}^{n/2} Be^{bj} < nBe^{bn/2} \ll p_n.$$

Combining this fact with the lower bound of Theorem 2.4 (2) implies that the proportion of nonprimitive conjugacy classes is at most

$$\frac{n^2 B}{A} e^{-bn/2}.$$

Hence, for large n, the nonprimitive conjugacy classes form a vanishingly small proportion of all conjugacy classes up to length n.

3. Cubical presentations and small-cancellation theory

This section reviews some definitions and results about cubical small-cancellation theory. Our primary reference is Wise [Wis21, Chapters 3–5]. We also prove Theorem 3.5, a properness criterion that follows from [Wis21, Thm 5.44] but is easier to apply.

3.1. **Cubical presentations and pieces.**

Definition 3.1 (Cubical presentation)**.** A *cubical presentation* $X^*=\langle X|Y_1,\ldots,Y_m\rangle$ consists of nonpositively curved cube complexes X and Y_i, and a set of local isometries $Y_i \looparrowright X$. The cubical presentation X^* corresponds to a topological space, also denoted X^*, consisting of X with a cone on each Y_i. Accordingly, we call each Y_i a *cone* of X^*.

See Figure 1 for an example. In this paper, it will always be the case that $\pi_1 Y_i \cong \mathbb{Z}$. However, this assumption is not present elsewhere in the literature.

For a hyperplane $\widetilde{U}$ of $\widetilde{X}$, the *carrier* $N(\widetilde{U})$ is the union of all closed cubes intersecting $\widetilde{U}$.

The *systole* $\|X\|$ is the infimal length of an essential combinatorial closed path in X. In terms of Definition 2.1, $\|X\|$ is the smallest translation length of a nontrivial element of $\pi_1 X$ acting on $\widetilde{X}$.

Definition 3.2 (Pieces)**.** Let $X^* = \langle X|Y_1,\ldots,Y_m\rangle$ be a cubical presentation. A *cone-piece* of X^* between Y_i and Y_j is a component of $\widetilde{Y}_i \cap \widetilde{Y}_j$, for some choice of lifts of $\widetilde{Y}_i$ and $\widetilde{Y}_j$ to $\widetilde{X}$, excluding the case where $\widetilde{Y}_i = \widetilde{Y}_j$.

A *wall-piece* of X^* in Y_i is a nonempty intersection of $\widetilde{Y}_i \cap N(\widetilde{U})$, where $\widetilde{U}$ is a hyperplane that is disjoint from $\widetilde{Y}_i$.

Given a constant $\alpha > 0$, we say that X^* satisfies the $C'(\alpha)$ *small-cancellation condition* if every cone-piece or wall-piece P involving Y_i satisfies $\mathrm{diam}(P) < \alpha\|Y_i\|$.

When pieces are small, the topological space X^* satisfies a number of pleasant properties.

Lemma 3.3 ([Wis21, Thm 3.32 and Thm 4.1])**.** *If X^* is $C'(\frac{1}{12})$, then every cone $Y_i \looparrowright X^*$ lifts to an embedding $Y_i \hookrightarrow \widetilde{X^*}$.*

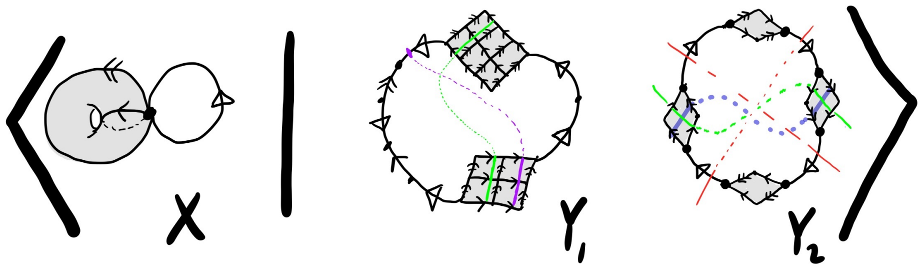

FIGURE 1. A cubical presentation illustrating conditions (1) and (2) of Definition 3.6. In this example, as in Theorem 3.5, we have $\pi_1 Y_i \cong \mathbb{Z}$ for each Y_i, and the wallspace structure on Y_i has two diametrically opposed hyperplanes in each wall. Unlike the setting of Theorem 3.5, $\pi_1 X$ is not hyperbolic in this example.

In a generalization of ordinary small-cancellation theory, small pieces guarantee the persistence of hyperbolicity in X^*.

Lemma 3.4 ([Wis21, Lem 3.70 and Thm 4.7])**.** *If $\pi_1 X$ is hyperbolic and X^* is compact and $C'(\frac{1}{14})$, then $\pi_1 X^*$ is hyperbolic.*

The main result of this section is Theorem 3.5, which guarantees that $\pi_1 X^*$ acts properly on a CAT(0) cube complex.

Theorem 3.5. *Let $X^* = \langle X \mid Y_1, \ldots, Y_k \rangle$ be a $C'(\frac{1}{20})$ cubical presentation. Suppose that X is compact, and every Y_i is compact and deformation retracts to a closed-geodesic. In addition, suppose that for every hyperplane $U \subset Y_i$, the carrier $N(U)$ is embedded, the complement $Y_i \smallsetminus U$ is contractible, and furthermore* $\operatorname{diam}(N(U)) < \frac{1}{20}\|Y_i\|$.

Then $\pi_1 X^*$ acts properly and cocompactly on a CAT(0) cube complex dual to a wallspace structure on $\widetilde{X^*}$ satisfying the $B(8)$ condition.

Moreover, if $\overline{g} \in \pi_1 X^* \smallsetminus \{1\}$ stabilizes a cell of the dual cube complex, then $\overline{g}$ is the image of an element $g \in \pi_1 X$ such that a conjugate of some $\pi_1 Y_i$ lies in $\langle g \rangle$. In particular, if each $\pi_1 Y_i$ is maximal cyclic, then $\pi_1 X^*$ acts freely on $\widetilde{X^*}$.

Theorem 3.5 will be proved with the aid of [Wis21, Thm 5.44 and Cor 5.45]. Setting up the proof using those results requires a number of auxiliary definitions, beginning with the $B(8)$ condition.

3.2. **Wallspace small-cancellation conditions.** We will construct a wallspace structure for $\widetilde{X^*}$. To do so, we will define a wallspace on each Y_i, whose walls are equivalence classes of hyperplanes in Y_i. This will generate a wallspace structure for $\widetilde{X^*}$ whose walls are equivalence classes of hyperplanes generated by the equivalence relation fostered by the wallspace structures on the lifts $Y_i \hookrightarrow \widetilde{X^*}$.

Definition 3.6 ($B(6)$ and $B(8)$ conditions)**.** Suppose that $X^* = \langle X \mid Y_1, \ldots, Y_k \rangle$ is a $C'(\frac{1}{12})$ cubical presentation. A $B(6)$ *structure on* X^* consists of a wallspace structure on each Y_i, which is preserved by its automorphisms, and satisfies a certain small-cancellation condition (see Figure 1 for an example). More precisely:

(1) The collection of hyperplanes of each Y_i is partitioned into classes satisfying the following conditions. No two hyperplanes in the same class cross or

osculate; in particular, the carrier of each hyperplane u_j embeds. The union $U = \cup u_j$ of the hyperplanes in a class is called a *wall*. Furthermore, for each wall U, there are *halfspaces* $\overleftarrow{U}, \overrightarrow{U}$ such that $Y_i = \overleftarrow{U} \cup \overrightarrow{U}$ and $U = \overleftarrow{U} \cap \overrightarrow{U}$.

(2) $\mathrm{Aut}(Y_i \to X)$ preserves the above *wallspace structure* on Y_i.

(3) If P is a path in Y_i that is the concatenation of at most 7 piece-paths and P starts and ends on the carrier $N(U)$ of a wall U, then P is path-homotopic into $N(U)$.

The $B(8)$ *condition* is defined by replacing (3) with the stronger condition:

(3′) If P is a path that is the concatenation of at most 8 piece-paths and P starts and ends on the carrier $N(U)$ of a wall then P is path-homotopic into $N(U)$.

In the above, a *piece-path* in Y is a path in a piece of Y. Meanwhile, $\mathrm{Aut}(Y_i \to X)$ is the group of automorphisms $\phi : Y_i \to Y_i$ such that
$$\begin{tikzcd} Y_i \arrow[rr] \arrow[dr] & & Y_i \arrow[dl] \\ & X & \end{tikzcd}$$
commutes.

3.3. **The properness criterion.** We now collect some terminology that will be needed to state and apply Theorem 3.8.

In our usage, geodesics are globally distance realizing. By contrast, a *closed-geodesic* $w \to Y$ in a nonpositively curved cube complex is a combinatorial immersion of a circle whose universal cover $\widetilde{w}$ lifts to a combinatorial geodesic $\widetilde{w} \to \widetilde{Y}$ in the universal cover of Y. We emphasize that (the image of) a closed-geodesic is not a geodesic in Y, because it is not distance realizing.

Let U be a hyperplane and v a 0-cube. We say that U is *m-proximate* to v if there is a path $P = P_1 \cdots P_m$ such that each P_i is either a single edge or a path in a piece, and v is the initial vertex of P_1 and U is dual to an edge in P_m. A wall is *m-proximate* to v if it has a hyperplane that is m-proximate to v.

A hyperplane u of a cone Y of X^* is *piecefully convex* if the following holds: for any path $\xi\rho \to Y$ with endpoints on $N(u)$, if ξ is a geodesic and ρ is either trivial or lies in a piece of Y containing an edge dual to u, then $\xi\rho$ is path-homotopic in Y to a path $\mu \to N(u)$.

We will verify pieceful convexity via the following criterion.

Remark 3.7 ([Wis21, Rem 5.43]). Let M be the maximal diameter of any piece of Y in X^*. Then a hyperplane u of Y is piecefully convex provided its carrier $N = N(u)$ satisfies $\mathsf{d}_{\widetilde{Y}}(g\widetilde{N}, \widetilde{N}) > M$ for any translate $g\widetilde{N} \neq \widetilde{N} \subset \widetilde{Y}$.

The following is a simplified restatement of [Wis21, Thm 5.44], which also incorporates [Wis21, Lem 3.70 and Cor 5.45].

Theorem 3.8. *Suppose that $X^* = \langle X \mid \{Y_i\} \rangle$ satisfies the following hypotheses:*

(1) *X^* is $C'(\frac{1}{14})$ and satisfies the $B(6)$ condition.*

(2) *Each hyperplane of each cone Y_i is piecefully convex.*

(3) *For each Y_i, each infinite order element of $\mathrm{Aut}(Y_i)$ is cut by a wall of Y_i.*

(4) *Let $\kappa \to Y_i$ be a geodesic with endpoints p, q. Let u_1 and u_1' be distinct hyperplanes in the same wall of Y_i. Suppose κ traverses a 1-cell dual to u_1, and either u_1' is 1-proximate to q or κ traverses a 1-cell dual to u_1'. Then there is a wall U_2 in Y_i that separates p, q but is not 2-proximate to p or q.*

Then $\pi_1 X^*$ acts with torsion stabilizers on the dual cube complex of the $B(6)$ structure.

We refer to Figure 2 for a depiction of the notation in hypothesis (4). We will now use Theorem 3.8 to prove Theorem 3.5.

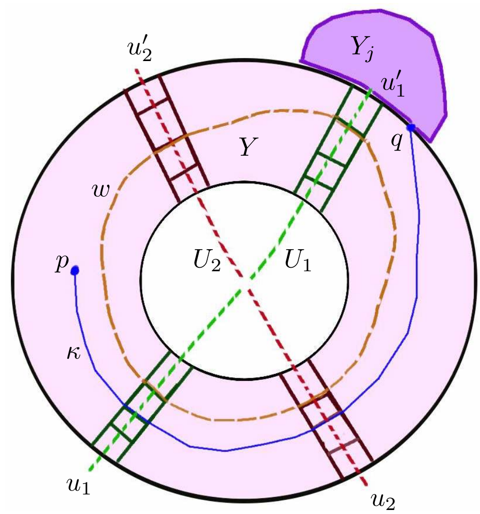

FIGURE 2. The scenario that arises in hypothesis (4) of Theorem 3.8. For the geodesic κ, we have to verify that neither p nor q is 2-proximate to $U_2 = u_2 \cup u_2'$.

Proof of Theorem 3.5. Observe that Definition 3.2 of the $C'(\alpha)$ condition involves a strict inequality. Since X^* is a finite $C'(\frac{1}{20})$ presentation, we may choose a constant $\alpha < \frac{1}{20}$ such that X^* is $C'(\alpha)$ and such that $\operatorname{diam} N(u) < \alpha \|Y_i\|$ for every hyperplane of every Y_i. Fix such an α for the remainder of the proof.

Observe that cubical subdivision preserves all the hypotheses of the theorem, and the $C'(\alpha)$ condition in particular. We subdivide X a number of times, while retaining the original metric. That is, every original edge of X continues to have length 1, while every edge of the m^{th} subdivision has length 2^{-m}. The reason for the iterated subdivision is that the length of an edge contributes additive error to several calculations below, and a large value of m will make this additive error negligible. For instance, since $\alpha < \frac{1}{20}$ and $\|Y_i\| \geq 1$, a large value of m ensures that

$$20\alpha \|Y_i\| < \|Y_i\| - 2^{-m} \quad \text{for every } i.$$

Other inequalities with additive constants will follow similarly.

After the first subdivision of X, the hyperplanes of X and of Y_i become 2-sided. By hypothesis, there is a closed-geodesic w_i that generates $\pi_1 Y_i$. Moreover, we may assume that $\operatorname{Aut}(Y_i \to X)$ stabilizes w_i, for the following reason. Since Y_i is compact, $\widetilde{Y}_i$ is a quasiline, hence $\operatorname{Stab}(\widetilde{Y}_i)$ is virtually cyclic. Let g be the generator of the maximal cyclic subgroup in $\operatorname{Stab}(\widetilde{Y}_i)$. Since we have subdivided X at least once, g stabilizes a geodesic axis $\widetilde{v}_i \subset \widetilde{Y}_i$, by [Hag23]. Let $v_i = \langle g \rangle \backslash \widetilde{v}_i$. Now, we may choose $w_i = \langle g^n \rangle \backslash \widetilde{v}_i$, where g^n generates $\pi_1(Y_i)$.

After the first subdivision, the closed-geodesic $w_i \subset Y_i$ has an even number of edges. The hypothesis that every hyperplane of Y_i has contractible complement implies that every hyperplane intersects some edge of w_i. Our wallspace structure on each Y_i will be defined by declaring each wall to consist of the pair of hyperplanes corresponding to a pair of antipodal edges of w_i. (See Figure 1 for a partial illustration of such a pairing, where the closed-geodesics w_i are presumed to run along the inner boundary of each Y_i.)

For the rest of the proof, we focus on one relator $Y = Y_i$ and its closed-geodesic $w = w_i$. Before verifying the hypotheses of Theorem 3.8, we check an inequality that will be very useful in the sequel.

Let $u, u' \subset Y$ be a pair of hyperplanes in the same wall of Y. This means that u, u' are dual to edges $\dot{u}, \dot{u}' \subset w$. Recall that u has an embedded carrier such that $\operatorname{diam}(N(u)) < \alpha\|Y\|$, and similarly for $N(u')$. Now, let $x \in N(u)$ and $x' \in N(u')$ be arbitrary. Then

$$\begin{aligned} \mathsf{d}(x,x') &\geq \mathsf{d}(\dot{u},\dot{u}') - \mathsf{d}(x,\dot{u}) - \mathsf{d}(\dot{u}',x') \\ &> \left(\frac{1}{2}\|Y\| - 2^{-m}\right) - 2\alpha\|Y\| \qquad (3.1) \\ &> 8\alpha\|Y\|, \qquad (3.2) \end{aligned}$$

because $\alpha < \frac{1}{20}$ and m is chosen so that 2^{-m} is tiny.

Next, we check the hypotheses of Theorem 3.8. In lieu of the $B(6)$ condition of Definition 3.6, we will in fact verify the stronger $B(8)$ condition.

To check condition (1) of Definition 3.6, let u, u' be distinct hyperplanes in the same wall of Y. Then each of $N(u)$ and $N(u')$ is embedded by hypothesis. Furthermore, equation (3.2) implies $N(u) \cap N(u') = \emptyset$. Thus $u \cup u'$ partition Y into two halfspaces. Condition (2) of Definition 3.6 holds because our (revised) choice of w ensures that $\operatorname{Aut}(Y \to X)$ stabilizes w, hence preserves the wallspace structure.

For condition (3′), consider $P = P_1 \cdots P_8$, a concatenation of 8 piece-paths that starts on $N(u)$ and ends on either $N(u')$ or on $N(u)$. We may assume that each P_i is a geodesic. If P ends on $N(u')$, then the $C'(\alpha)$ hypothesis implies that $|P_i| < \alpha\|Y\|$, hence $|P| < 8\alpha\|Y\|$, contradicting equation (3.2).

Now, consider the case where P starts and ends on $N(u)$. Since we have already checked that P is too short to reach $N(u')$, it follows that P lifts to $\widetilde{Y} - (\pi_1 Y)\widetilde{u}'$, each of whose components is convex. Thus P is path-homotopic into $N(u)$. We have verified condition (3′) of Definition 3.6. Hence X^* satisfies the $B(8)$ condition, and hypothesis (1) of Theorem 3.8.

Hypothesis (2) of Theorem 3.8, namely pieceful convexity of the hyperplanes of Y, follows from Remark 3.7. Indeed, let $x \in N(\widetilde{u})$ and $x' \in gN(\widetilde{u})$ be points on the carriers of two distinct lifts of u to $\widetilde{Y}$. Then, by the same calculation as in equations (3.1) and (3.2), we have $\mathsf{d}(x,x') > 8\alpha\|Y\|$, which exceeds the diameter of a piece.

Hypothesis (3) of Theorem 3.8 holds vacuously, since each Y is compact.

We now verify hypothesis (4) of Theorem 3.8. Let $\kappa \to Y$ be a geodesic from p to q, as in that condition. We begin by deriving upper and lower bounds on $|\kappa| = \mathsf{d}(p,q)$. Let $\dot{p}, \dot{q}$ be vertices of w that are closest to p, q, respectively. Since a hyperplane carrier containing p cuts w and has diameter less than $\alpha\|Y\|$, we have $\mathsf{d}(p,\dot{p}) \leq \alpha\|Y\|$, and similarly $\mathsf{d}(q,\dot{q}) \leq \alpha\|Y\|$. Since $|w| = \|Y\|$, we have

$\mathsf{d}(\dot{p}, \dot{q}) \leq \frac{1}{2}\|Y\|$. Thus

$$\begin{aligned} |\kappa| = \mathsf{d}(p, q) &\leq \mathsf{d}(p, \dot{p}) + \mathsf{d}(\dot{p}, \dot{q}) + \mathsf{d}(\dot{q}, q) \\ &< \frac{1}{2}\|Y\| + 2\alpha\|Y\|. \end{aligned} \tag{3.3}$$

For the lower bound on $|\kappa|$, recall that there is a wall $U_1 = u_1 \cup u_1'$ such that κ traverses a 1-cell dual to u_1, and either u_1' is 1-proximate to q or κ traverses a 1-cell dual to u_1'. See Figure 2. If κ intersects u_1', then equation (3.1) implies

$$|\kappa| \geq \mathsf{d}(N(u_1), N(u_1')) > \left(\frac{1}{2}\|Y\| - 2^{-m}\right) - 2\alpha\|Y\|.$$

Otherwise, if q is 1-proximate to u_1', then $\mathsf{d}(q, u_1') < \alpha\|Y\|$, hence we obtain the weaker conclusion

$$\begin{aligned} |\kappa| &> \mathsf{d}(N(u_1), N(u_1')) - \alpha\|Y\| \\ &> \left(\frac{1}{2}\|Y\| - 2^{-m}\right) - 3\alpha\|Y\|. \end{aligned} \tag{3.4}$$

Now, let y be the midpoint of κ, and let u_2 be a hyperplane such that $y \in N(u_2)$. Let u_2' be the other hyperplane in the same wall U_2. We claim that neither u_2 nor u_2' is 2-proximate to p or q. To see this, we will check that $\mathsf{d}(\{p, q\}, u_2) > 2\alpha\|Y\|$, and similarly for u_2'. For any $x \in N(u_2)$, we have

$$\begin{aligned} \mathsf{d}(\{p, q\}, x) &\geq \mathsf{d}(\{p, q\}, y) - \mathsf{d}(x, y) \\ &> \frac{1}{2}|\kappa| - \alpha\|Y\| \\ &> \frac{1}{2}\left(\frac{1}{2}\|Y\| - 2^{-m} - 3\alpha\|Y\|\right) - \alpha\|Y\| \\ &= \frac{1}{4}\|Y\| - \frac{5}{2}\alpha\|Y\| - 2^{-(m+1)} \\ &> 2\alpha\|Y\|. \end{aligned}$$

Here, the first inequality is the triangle inequality, the second inequality is the diameter bound on $N(u_2)$, the third inequality is equation (3.4), and the final inequality follows because $\alpha < \frac{1}{20}$ and 2^{-m} is tiny. Similarly, for any $x' \in N(u_2')$, we have

$$\begin{aligned} \mathsf{d}(\{p, q\}, x') &\geq \mathsf{d}(x', y) - \mathsf{d}(\{p, q\}, y) \\ &> \left(\frac{1}{2}\|Y\| - 2^{-m} - 2\alpha\|Y\|\right) - \frac{1}{2}|\kappa| \\ &> \left(\frac{1}{2}\|Y\| - 2^{-m} - 2\alpha\|Y\|\right) - \frac{1}{2}\left(\frac{1}{2}\|Y\| + 2\alpha\|Y\|\right) \\ &= \frac{1}{4}\|Y\| - 3\alpha\|Y\| - 2^{-m} \\ &> 2\alpha\|Y\|. \end{aligned}$$

Here, the first inequality is the triangle inequality, the second inequality is equation (3.1), the third inequality is equation 3.3, and the final inequality holds because $\alpha < \frac{1}{20}$ and 2^{-m} is tiny. Thus neither p nor q can be 2-proximate to u_2 or u_2'. We have thus verified that Theorem 3.8 applies, hence $\pi_1 X^*$ acts with torsion stabilizers on the CAT(0) cube complex dual to the wallspace structure we have constructed. Since X is compact, this action is cocompact.

Finally, we check the conclusion about cell stabilizers in $\pi_1 X^*$. We have already shown that cell stabilizers must be torsion. By [Wis21, Thm 4.2 and Rem 4.3], we know that when $\alpha < \frac{1}{20}$, any torsion element in $\pi_1 X^*$ lies in $\mathrm{Aut}(Y_i \to X)$ for some lift $Y_i \hookrightarrow \widetilde{X^*}$. Recall that $\widetilde{Y_i}$ is quasiisometric to a line. If $\sigma \in \mathrm{Aut}(Y_i \to X)$ is nontrivial, let $\overline{Y_i} = \langle\sigma\rangle\backslash Y_i$. Then $\overline{Y_i} \to X$ is a local isometry, with $\pi_1(\overline{Y_i})$ an infinite cyclic subgroup properly containing $\pi_1 Y_i$. In particular, if $\pi_1 Y_i$ is maximal cyclic, then we obtain a contradiction, hence $\pi_1 X^*$ acts freely. □

4. Controlling pieces in generic quotients

This section contains the proof of the main theorem, Theorem 1.1. In the proof, we need to control the sizes of pieces in a generic cubical presentation. Wall-pieces are controlled in Proposition 4.7 and cone-pieces are controlled in a sequence of lemmas, culminating in Proposition 4.17.

Most of the work in this section occurs in the language of *loose pieces*, which represent a coarsening of the overlaps defined in Definition 3.2. We define loose pieces in Definition 4.4. In Lemma 4.5, we show that every wall-piece or cone-piece in the sense of Definition 3.2 gives rise to a corresponding loose piece.

Via loose pieces, the proofs of Propositions 4.7 and 4.17 readily generalize to the context of noncubical hyperbolic metric spaces in Section 5.

4.1. **Loose pieces and convex hulls of quasi-axes.**

Let Υ be a metric space. For any subset $s \subset \Upsilon$ and $J > 0$, the notation $\mathcal{N}_J(s)$ denotes the closed J-neighborhood of s.

Definition 4.1 (Quasi-axes)**.** Let $\widetilde{w}, \widetilde{w}'$ be bi-infinite geodesics in a metric space Υ. We declare them to be equivalent, and write $\widetilde{w} \approx \widetilde{w}'$, whenever $\widetilde{w} \subset \mathcal{N}_r(\widetilde{w}')$ for some $r \geq 0$. In the main case of interest, when Υ is δ-hyperbolic, we can use a uniform value $r = 2\delta$, and deduce that $\widetilde{w} \approx \widetilde{w}'$ if and only if $\partial\widetilde{w} = \partial\widetilde{w}'$.

Let g be a hyperbolic isometry of Υ. A *quasi-axis* for g is a bi-infinite geodesic $\widetilde{w} \subset \Upsilon$ such that $g\widetilde{w} \approx \widetilde{w}$. When Υ is δ-hyperbolic, any geodesic connecting the fixed points of g on $\partial\Upsilon$ is a quasi-axis for g.

Lemma 4.2. *Let $\widetilde{X}$ be a δ-hyperbolic, finite dimensional CAT(0) cube complex. Then there is a uniform constant $K = K(\widetilde{X})$ with the following property. For every bi-infinite geodesic $\widetilde{w}$, the convex hull of $\widetilde{w}$ satisfies*

$$\mathrm{hull}\,(\widetilde{w}) \subset \mathcal{N}_K(\widetilde{w}).$$

Proof. Consider a vertex $p \in \widetilde{X}$ that lies far from $\widetilde{w}$. Let s be a shortest CAT(0) (noncombinatorial) geodesic segment from p to $\widetilde{w}$, oriented outward from p. Sageev and Wise have observed [SW15, Remark 3.2] that the first cube met by s contains a midcube of a hyperplane H that intersects s at an angle bounded away from 0. The angle bound depends only on the dimension of the cube, hence can be taken uniformly over $\widetilde{X}$.

We claim that when s is sufficiently long, $H \cap \widetilde{w} = \emptyset$. In other words, H separates p from $\widetilde{w}$, hence $p \notin \mathrm{hull}(\widetilde{w})$. The claim holds because an intersection $H \cap \widetilde{w}$ would determine a CAT(0) geodesic triangle with a long base along s and two large angles along s. Such a triangle cannot be δ-thin. □

The following construction relates cubical presentations to quasi-axes. Given $G = \pi_1 X$ and $g \in G - \{1\}$, choose any quasi-axis $\widetilde{w}$ for g. Let $\widetilde{Y}$ be the convex hull of the union of all geodesics equivalent to $\widetilde{w}$. Then the quotient $Y = \langle g \rangle \backslash \widetilde{Y}$ admits a local isometry to X. By Lemma 4.2, $\widetilde{Y}$ lies in a uniform neighborhood of $\widetilde{w}$, hence its quotient Y is compact. We say that Y is *a quasicircle defined by g*.

If g stabilizes an axis $\widetilde{w}$, as will be the case following Convention 4.3, then the quasicircle Y deformation retracts to the closed-geodesic $w = \langle g \rangle \backslash \widetilde{w}$.

Now, we adopt Convention 4.3.

Convention 4.3. From now until the start of the proof of Theorem 1.1, the symbol X denotes a compact nonpositively curved cube complex, such that $\widetilde{X}$ is δ-hyperbolic for some δ. We assume that $G = \pi_1 X$ is nonelementary.

We assume that X has been subdivided at least once, so that every $g \neq 1$ stabilizes a geodesic axis in $\widetilde{X}$ [Hag23].

Finally, we assume that every hyperplane of X is essential. This assumption is harmless, because every nonpositively curved cube complex X' contains a convex essential core, whose hyperplanes are essential. See, for instance, Caprace and Sageev [CS11, Proposition 3.5].

Our arguments in this section will employ the following, looser analogue of Definition 3.2.

Definition 4.4 (Loose pieces)**.** Let X be as in Convention 4.3. Fix a constant $J \in \mathbb{N}$. We define J–loose pieces first in $\widetilde{X}$, then in X.

Consider bi-infinite geodesics $\widetilde{w}, \widetilde{w}'$ that do not share an endpoint in $\partial \widetilde{X}$. Then a *J–loose cone-piece between $\widetilde{w}$ and $\widetilde{w}'$* is the maximal geodesic segment $s \subset \widetilde{w}$ whose endpoints are contained in $\mathcal{N}_J(\widetilde{w}')$. The *companion of s* is the maximal segment $s' \subset \widetilde{w}'$ whose endpoints are at distance J from the corresponding endpoints of s.

Now, let w, w' be closed-geodesics in X. If $w \neq w'$, a *J–loose cone-piece* between w and w' is the projection of a J–loose cone-piece between preimages $\widetilde{w}$ and $\widetilde{w}'$. A *J–loose cone-piece between w and itself* arises in the above scenario where $\widetilde{w}' = h\widetilde{w}$ for some $h \notin \mathrm{Stab}_G(\widetilde{w})$.

In a similar manner, a *J–loose wall-piece in $\widetilde{w}$* is a maximal geodesic segment $s \subset \widetilde{w}$ whose endpoints are contained in $\mathcal{N}_J(N(\widetilde{U}))$ for a hyperplane U. A *J–loose wall-piece in a closed-geodesic w* is the projection of a J–loose wall-piece in $\widetilde{w}$.

Lemma 4.5 shows that there is a uniform value $J = J(\widetilde{X})$ such that every piece in a cubical presentation X^* must correspond to a J–loose piece in X.

Lemma 4.5. *Consider a cubical presentation $X^* = \langle X \mid Y_1, \ldots, Y_m \rangle$. Suppose that $\widetilde{X}$ is hyperbolic and that every Y_i is a quasicircle. Let $\widetilde{w}_i$ be a bi-infinite geodesic in $\widetilde{Y}_i$. Then there is a constant $J = J(\widetilde{X})$ such that the following hold for every $d \geq 0$:*

(1) *Every diameter $\geq d$ cone-piece of X^* between Y_i and Y_j determines a J–loose cone-piece between $\widetilde{w}_i$ and $\widetilde{w}_j$, of diameter $\geq d$.*
(2) *Every diameter $\geq d$ wall-piece of X^* in Y_i determines a J–loose wall-piece in $\widetilde{w}_i$, also of diameter $\geq d$.*
(3) *Every diameter $\geq d$ intersection between $\widetilde{Y}_i$ and a hyperplane carrier $N(U)$ determines a J–loose wall-piece in $\widetilde{w}_i$, of diameter $\geq d$.*

Furthermore, we may take $J \leq 3K$, where K is the constant of Lemma 4.2.

Proof. First, consider cone-pieces. Let $p, q \in \widetilde{Y}_i \cap \widetilde{Y}_j$ be points such that $\mathsf{d}(p, q) \geq d$. By Lemma 4.2, there are points $p_i, q_i \in \widetilde{w}_i$ that are K-close to p and q, respectively. Applying Lemma 4.2 to the quasicircle Y_j shows that p_i, q_i are $2K$-close to $\widetilde{w}_j$. Since $\widetilde{w}_i$ is a geodesic, we have $\mathsf{d}(p_i, q_i) \geq d - 2K$. Extending the segment $[p_i, q_i]$ by K in each direction, we obtain a geodesic segment $[p_i', q_i'] \subset \widetilde{w}_i$ whose length is at least d and whose endpoints lie in $\mathcal{N}_{3K}(\widetilde{w}_j)$. Thus $[p_i, q_i]$ is contained in a $3K$–loose cone-piece between $\widetilde{w}_i$ and $\widetilde{w}_j$.

The proof for wall-pieces and intersections with hyperplanes is identical. In this case, we may take $J = 2K$. $\square$

Remark 4.6. Besides the looseness constant J, there is one important respect in which Definition 4.4 is more general than Definition 3.2. Whereas the definition of a wall-piece in Y_i requires the hyperplane U to be disjoint from Y_i, the definition of a J–loose wall-piece in $\widetilde{w}_i$ has no analogous restriction. Consequently, the J–loose wall-pieces in Lemma 4.5.(3) need not come from wall-pieces. When we apply Theorem 3.5, we will care about *all* intersections between Y_i and hyperplane carriers in X, and all such intersections will be controlled using J–loose wall-pieces.

4.2. **Wall-pieces.** Our next goal is to give a criterion on the growth of the number of relators ensuring that with overwhelming probability, the diameter of wall-pieces in a cubical presentation is bounded by $\alpha\|Y_i\|$. We formulate the criterion in terms of loose pieces, for greater applicability in Theorem 3.5 and in the generalized setting of Section 5.

Recall that by Convention 4.3, we are assuming that G is nonelementary and the hyperplanes of X are essential. By Theorem 2.4, the growth function of G acting on $\widetilde{X}$ is bounded by constants times $e^{b\ell}$, where $b > 0$ is the growth exponent. Let a be the largest growth exponent among the carriers of the finitely many orbits of hyperplanes of $\widetilde{X}$. Then a is also the largest growth exponent of any of the J-neighborhoods of the hyperplanes, for an arbitrary J. Recall that $a < b$ by Theorem 2.3.

For each $\ell > 0$, let $\mathcal{G}(\ell)$ denote the set of nontrivial conjugacy classes in $\pi_1 X$ of length at most ℓ. For the duration of this section, we will be sampling randomly from $\mathcal{G}(\ell)$, for large ℓ.

Proposition 4.7. *Let X be as in Convention* 4.3*. For each nontrivial conjugacy class $\pi_1 X$, choose a representative closed-geodesic. Let $a < b$ be growth exponents as above, let $\alpha \in (0, 1)$, and fix a positive number $c < \alpha(b - a)$. Then, for every $J \in \mathbb{N}$, there is a constant $C = C(J, X)$, with the following property.*

For $k \leq e^{c\ell}$, choose conjugacy classes $[g_1], \ldots, [g_k] \in \mathcal{G}(\ell)$ uniformly at random. Let w_i be the closed-geodesic representing $[g_i]$. Then, for all $\ell \gg 0$, the probability that some w_i contains a J–loose wall-piece of diameter at least $\alpha\|w_i\|$ is bounded above by

$$C\ell^2 e^{(c+(a-b)\alpha)\ell}.$$

Consequently, with overwhelming probability as $\ell \to \infty$, all J–loose wall-pieces in w_i have diameter less than $\alpha|g_i|$.

Proof. Let $\Pi(\alpha, \ell)$ denote the proportion of representative closed-geodesics w of length at most ℓ such that w contains a J–loose wall-piece of diameter at least $\alpha\|w\|$. That is, $\Pi(\alpha, \ell)$ denotes the proportion of all w of length at most ℓ such that

$\widetilde{w} \cap \mathcal{N}_J(N(\widetilde{U}))$ contains the endpoints of a segment of length at least $\alpha\|w\|$ for some hyperplane U. Our goal is to estimate $\Pi(\alpha, \ell)$.

For each $n \leq \ell$, let $\mathcal{U}(n, \alpha)$ be the set of (conjugacy representatives of) closed-geodesics w of length exactly n that contain a J–loose wall-piece s with $|s| \geq \alpha n$. Then

$$\Pi(\alpha, \ell) = \frac{\sum_{n=1}^{\ell} |\mathcal{U}(n, \alpha)|}{|\mathcal{G}(\ell)|}.$$

Each element $w \in \mathcal{U}(n, \alpha)$ can be cyclically permuted to be of the form $s \cdot y$, where s lies in $\mathcal{N}_J(N(\widetilde{U}))$ for some hyperplane $\widetilde{U}$, and $|s| \geq \alpha n$. By Theorem 2.4 (1) and Remark 2.5, the number of such paths s, up to homotopy in $\widetilde{X}$, is bounded above by $B' e^{a|s|}$, for some constant B' depending on J. Similarly, the number of paths y is bounded above by $B'' e^{b|y|}$, for some constant B''.

Let $B = B'B''$. Since there are at most n distinct cyclic permutations of w, the number of elements in $\mathcal{U}(n, \alpha)$ is

$$|\mathcal{U}(n, \alpha)| \leq n \cdot B e^{a|s|} e^{b|y|} = Bn\, e^{a|s| + b(|w| - |s|)} \leq Bn\, e^{(a\alpha + b(1-\alpha))n}. \tag{4.1}$$

Summing over all lengths up to ℓ, we obtain

$$\begin{aligned}
\sum_{n=1}^{\ell} |\mathcal{U}(n, \alpha)| &\leq \sum_{n=1}^{\ell} Bn e^{(a\alpha + b(1-\alpha))n} \\
&\leq B\ell \sum_{n=1}^{\ell} \left(e^{a\alpha + b(1-\alpha)}\right)^n \\
&< B\ell \sum_{m=0}^{\infty} \left(e^{a\alpha + b(1-\alpha)}\right)^{\ell - m} \\
&= B\ell \, \frac{e^{(a\alpha + b(1-\alpha))\ell}}{1 - e^{-(a\alpha + b(1-\alpha))}} \\
&= e^{b\ell} \cdot \ell e^{(a-b)\alpha\ell} \cdot \frac{B}{1 - e^{-(a\alpha + b(1-\alpha))}} \\
&\leq e^{b\ell} \cdot \ell e^{(a-b)\alpha\ell} \cdot \frac{B}{1 - e^{-a}}.
\end{aligned}$$

Meanwhile, Theorem 2.4 (2) implies that $|\mathcal{G}(\ell)| \geq \frac{A}{\ell} e^{b\ell}$. Thus

$$\Pi(\alpha, \ell) \leq \frac{\sum |\mathcal{U}(n, \alpha)|}{|\mathcal{G}(\ell)|} \leq \ell^2 e^{(a-b)\alpha\ell} \cdot \frac{B}{A(1 - e^{-a})}. \tag{4.2}$$

Among all choices $(w_1, w_2, \ldots, w_k) \in \mathcal{G}(\ell)^k$, the proportion of k-tuples where some $w_i \in \mathcal{U}(n, \alpha)$ is bounded above by $k\Pi(\alpha, \ell)$. Since $k \leq e^{c\ell}$, equation (4.2) says that the total probability of a large piece is bounded by

$$k\Pi(\alpha, \ell) \leq \ell^2 e^{c\ell + (a-b)\alpha\ell} \cdot \frac{B}{A(1 - e^{-a})}.$$

Setting $C = \frac{B}{A(1-e^{-a})}$ completes the proof of the estimate. In particular, since $c + (a - b)\alpha < 0$ by hypothesis, we have $k\Pi(\alpha, \ell) \to 0$ as $\ell \to \infty$. $\square$

4.3. **Cone-pieces between two closed-geodesics.** Our next goal is to control cone-pieces whose diameter is bounded below by a constant d. The gist of the following sequence of lemmas is that the probability of a J–loose cone-piece of diameter at least d declines exponentially with d. See Proposition 4.14.

We will need to employ separate arguments for pieces between closed-geodesics w and w' chosen separately, and pieces between a closed-geodesic w and itself. In the case of a piece between w and itself, we will further need to consider overlaps, defined as follows.

Definition 4.8 (Orientation and overlaps)**.** Let $w \to X$ be a closed-geodesic. Choose distinct preimages $\widetilde{w}$ and $h\widetilde{w}$ of w and a constant $J \in \mathbb{N}$. Suppose $s = [p, q] \subset \widetilde{w}$ is a J–loose piece between $\widetilde{w}$ and $h\widetilde{w}$, such that $|s| > 2J$. Let $s' = [p', q'] \subset h\widetilde{w}$ be the companion of s, so that $\mathsf{d}(p, p') = J = \mathsf{d}(q, q')$.

Suppose that $\widetilde{w}$ is oriented from p to q, and transfer this orientation to $h\widetilde{w}$ via h. Since $|s| = \mathsf{d}(p, q) > 2J$, the endpoints p', q' of the companion s' cannot coincide. We say that h *preserves orientation on* s if p' comes before q' in the orientation on $h\widetilde{w}$, and *reverses orientation on* s otherwise. See Figure 3.

The h-*overlap in* s is the interval $u = s \cap h^{-1}s'$.

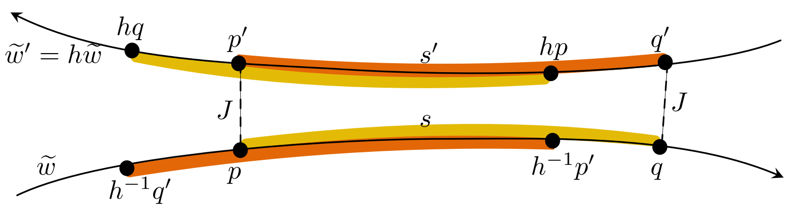

FIGURE 3. The terminology and setup of Definition 4.8. The J–loose piece $s = [p, q]$ is shown in yellow, and the companion $s' = [p', q']$ in orange. In this example, h reverses orientation on the piece s, and s has nontrivial h-overlap $[p, h^{-1}p']$.

The main steps of the proof of Proposition 4.14 can be organized as follows:

- In Lemma 4.10, we show that any orientation-reversing piece between w and itself has small overlap, universally bounded by a constant R. This is an unconditional (nonprobabilistic) statement.
- In Lemma 4.11, we show that pieces between separately sampled closed-geodesics w and w' are exponentially rare. Similarly, pieces between w and itself with small overlap (of size $\leq R$) are exponentially rare.
- In Lemma 4.13, we show that pieces between w and itself with big overlap (of size $\geq R$) are also exponentially rare. This allows us to conclude that each type of piece of diameter $\geq d$ is exponentially rare.

We will need the following standard fact about thin quadrilaterals.

Lemma 4.9. *Suppose that $\widetilde{X}$ is δ-hyperbolic. Let Q be a geodesic quadrilateral with corners p, q, q', p', such that $\mathsf{d}(p, p') \leq J$ and $\mathsf{d}(q, q') \leq J$. Then every point $x \in [p, q]$ lies within distance $\mathring{J} = J + 2\delta$ of some point of $[p', q']$.*

Proof. This is standard. Since triangles in $\widetilde{X}$ are δ-thin, then quadrilaterals are 2δ-thin. Thus $x \in [p, q]$ is 2δ-close to some point $x' \in [p, p'] \cup [p', q'] \cup [q', q]$. But every point of $[p, p'] \cup [q', q]$ is J-close to $[p', q']$. $\square$

Lemma 4.10. *Suppose X is compact and $\widetilde{X}$ is δ-hyperbolic. For every $J \in \mathbb{N}$, there is a constant $R = R(J, \delta) \geq 10J$ such that the following holds. Suppose that $s \subset \widetilde{w}$ is a J–loose cone-piece between $\widetilde{w}$ and $h\widetilde{w}$, where h reverses orientation on s. Then the h-overlap $u \subset s$ has diameter $|u| \leq R$.*

Proof. In the notation of Definition 4.8, the overlap is $u = [p, q] \cap h^{-1}[p', q']$. Since h is orientation-reversing, the endpoints of u are $x \in \{p, h^{-1}q'\}$ and $y \in \{q, h^{-1}p'\}$. We begin by showing there is a constant $L = L(J, \delta)$ such that

$$\mathsf{d}(x, hy) \leq L \quad \text{and} \quad \mathsf{d}(y, hx) \leq L. \tag{4.3}$$

To prove (4.3), we consider two cases. For the first case, suppose $\mathsf{d}(p, h^{-1}q') \leq 2J$ and $\mathsf{d}(q, h^{-1}p') \leq 2J$. Since $x \in \{p, h^{-1}q'\}$ and $hy \in \{p', hq\}$, triangle inequalities imply that

$$\mathsf{d}(x, hy) \leq \mathsf{d}(x, p) + \mathsf{d}(p, p') + \mathsf{d}(p', hy) \leq 2J + J + 2J = 5J.$$

By an identical calculation, $\mathsf{d}(y, hx) \leq 5J$.

For the second case, suppose that $\mathsf{d}(p, h^{-1}q') > 2J$ or $\mathsf{d}(q, h^{-1}p') > 2J$. Since we have $|\mathsf{d}(p, q) - \mathsf{d}(p', q')| \leq 2J$, it follows that one of $h^{-1}(q'), h^{-1}(p')$ lies in $[p, q]$. Assume without loss of generality that $h^{-1}(p') \in [p, q]$. If we furthermore assume that $\mathsf{d}(q, h^{-1}p') > 2J$, then the endpoints of u are $x = p$ and $y = h^{-1}p'$, as depicted in Figure 3. It follows immediately that

$$\mathsf{d}(x, hy) = \mathsf{d}(p, p') = J.$$

Furthermore, as in Lemma 4.9, every point of $[p, q]$ is 2δ-close to some point of $[p, p'] \cup [p', q'] \cup [q', q]$. Since $\mathsf{d}(y, q) > J$, it follows that $y \in [p, q]$ is 2δ-close to some point $y' \in [p', q']$. Furthermore, since $\mathsf{d}(x, y) = \mathsf{d}(hy, hx) = \mathsf{d}(p', hx)$, triangle inequalities imply that $\mathsf{d}(hx, y') \leq J + 2\delta$. Thus

$$\mathsf{d}(y, hx) \leq \mathsf{d}(y, y') + \mathsf{d}(y', hx) \leq 2\delta + (J + 2\delta) = J + 4\delta.$$

It remains to consider the possibility that $h^{-1}(p') \in [p, q]$ and $\mathsf{d}(q, h^{-1}p') \leq 2J$. Since we are in the second case, triangle inequalities imply that $2J < \mathsf{d}(p, h^{-1}q') \leq 4J$. Now, we argue exactly as in the first case, and conclude

$$\mathsf{d}(x, hy) \leq 4J + J + 2J = 7J \quad \text{and} \quad \mathsf{d}(y, hx) \leq 4J + J + 2J = 7J.$$

Setting $L = \max(7J, J + 4\delta)$ completes the proof of equation (4.3).

Now, we complete the proof of the lemma. Applying (4.3) twice shows that $\mathsf{d}(x, h^2x) \leq 2L$ and $\mathsf{d}(h^{-1}x, hx) \leq 2L$. By Convention 4.3, h has an invariant geodesic axis $\widetilde{v}$ in $\widetilde{X}$, hence $[\![h]\!] \geq 1$. Thus the $\langle h^2 \rangle$-orbit $\{h^{2n}(x) \mid n \in \mathbb{Z}\}$ lies along a quasigeodesic whose quality depends only on J and δ. Similarly, $\{h^{2n}(hx) \mid n \in \mathbb{Z}\}$ lies along a uniform quasigeodesic.

The above quasigeodesics must uniformly fellow-travel $\widetilde{v}$. As a consequence, there is a uniform radius r, depending only on J and δ, such that $\langle h \rangle(x) \subset \mathcal{N}_r(\widetilde{v})$.

Since $x, hx \in \mathcal{N}_r(\widetilde{v})$, considering closest-point projections to $\widetilde{v}$ gives

$$\mathsf{d}(x, hx) \leq r + [\![h]\!] + r \leq 2r + L.$$

Meanwhile equation (4.3) gives $\mathsf{d}(hx, y) \leq L$. Thus $R = 2r + 2L$ is a uniform bound on $|u| = \mathsf{d}(x, y)$. Since $L \geq 7J$ by definition, we have $R \geq 14J$. $\square$

Now, we argue that closed-geodesics containing a piece of diameter $\geq d$ happen with a probability that decays exponentially with d.

Lemma 4.11. *Suppose that every nontrivial element of $\pi_1 X$ stabilizes a geodesic in a δ-hyperbolic space $\widetilde{X}$. For each nontrivial conjugacy class in $\pi_1 X$, choose a closed-geodesic in X that represents it.*

Fix constants $J, R \in \mathbb{N}$. Then, for all $d > 2J$ and all ℓ sufficiently large, the following hold:

1. *Among all pairs of conjugacy classes $[g], [g']$ of length at most ℓ, the proportion whose representative closed-geodesics have a J–loose cone-piece of diameter $\geq d$ is less than $M\ell^2\, e^{-bd}$, for a constant $M = M(X, J)$.*
2. *Among all conjugacy classes $[g]$ of length at most ℓ, the proportion whose representative closed-geodesic has a J–loose cone-piece with itself, with diameter $\geq d$ and overlap of length $\leq R$, is less than $M_0\ell^2 e^{-bd}$, for a constant $M_0 = M_0(X, J, R)$.*

Proof. Before beginning the probabilistic portion of the proof, we make a reduction, introduce notation, and name some constants. Let D be the diameter of X. Let $\mathring{J} = J + 2\delta$, and let V be the maximal number of vertices in a ball of radius $(D + J + \mathring{J})$ in $\widetilde{X}$. Let W be the maximal number of vertices in a ball of radius n_0, where n_0 is the threshold constant of Theorem 2.4. Fix a basepoint $x_0 \in \widetilde{X}$.

Suppose that $[g], [g']$ are conjugacy classes represented by closed-geodesics w, w' that have a J–loose cone-piece. (For now, we do not place any constraint on the lengths of w, w', or the piece.) By Definition 4.4, the piece in w is the image of a geodesic segment $s \subset \widetilde{w} \subset \widetilde{X}$, which is a J–loose cone-piece between $\widetilde{w}$ and $\widetilde{w}'$. By choosing the preimage s appropriately, we ensure the additional property that the initial point $p \in s$ lies within radius D of the basepoint $x_0 \in \widetilde{X}$. Let $s' \subset \widetilde{w}'$ be the companion of s. Then the initial point of s' is a point p' that lies J-close to $p \in S$, hence within distance $D + J$ of the basepoint x_0. Furthermore, by Lemma 4.9, every point of s' lies within distance $\mathring{J} = J + 2\delta$ of some point of s.

Since conjugacy classes in $\pi_1 X$ correspond to free homotopy classes in X, every geodesic segment in $\widetilde{w}'$ of length $|w'|$ is a fundamental domain for w'. It will be convenient to choose a fundamental domain $v' \subset \widetilde{w}'$ that begins at p', such that the initial segment of v' coincides with the initial segment of s'. Since the orientation of s' (determined by the condition that the initial point $p' \in s'$ is J-close to $p \in s$) may or may not coincide with the orientation of w', we conclude that the segment v' chosen as above is a fundamental domain for $(w')^{\pm 1}$.

Now, we proceed to the proof of Lemma 4.11 (1). Fix a conjugacy class $[g]$ and its representative closed-geodesic w. We will bound the number of conjugacy classes $[g']$, with representative closed-geodesic w', such that w has a J–loose cone-piece with w', of diameter at least d.

As in the above notation, let $s \subset \widetilde{w}$ be a preimage of the piece in w that begins within distance D of the basepoint x_0. As in the opening paragraph of the proof, let $s' \subset \widetilde{w}'$ be the corresponding subsegment of $\widetilde{w}'$, so that the initial point $p \in s$ is J-close to the initial or terminal point $p' \in s'$.

As a warm-up case, suppose that $|s'| \geq |w'| - n_0$, where n_0 is the threshold constant in Theorem 2.4. As above, $(w')^{\pm 1}$ has a fundamental domain $v' \subset \widetilde{w}'$ whose initial segment coincides with the initial segment of s'. More precisely, we have a fundamental domain $v' = [p', r']$ and a point $q' \in [p', r']$ such that $s' \cap v' = [p', q']$ and $\mathsf{d}(q', r') \leq n_0$. Then, by construction, we have $\mathsf{d}(p, p') = J$.

Furthermore, by Lemma 4.9, we have $\mathsf{d}(q', q) \le \mathring{J}$ for some point $q \in s$. Observe that $|\mathsf{d}(p, q) - \mathsf{d}(p', q')| \le J + \mathring{J}$.

By the definition of V, and the choice of s so that its start lies close to the basepoint, there are at most V choices for where p' can lie. Then, once $n = \mathsf{d}(p, q)$ is chosen, the point $q \in s$ is determined, hence there are at most V choices for $q' \in s'$. Since $\mathsf{d}(q', r') \le n_0$, it follows that once q' is chosen, there are at most W choices for r'. Consequently, for every n, there are at most V^2W choices for the segment v', hence at most $2V^2W$ choices for $[g']$, where the factor of 2 accounts for the orientation of g'. Recalling that $n \le \ell + J + \mathring{J}$, the total number of possibilities for $[g']$ is at most

$$2V^2W \cdot (\ell + J + \mathring{J}).$$

In particular, the bound grows linearly with ℓ in the (nongeneric) warm-up case.

Having finished the warm-up case, assume that $|s'| < |w'| - n_0$. Then, as above, $(w')^{\pm 1}$ has a fundamental domain $v' \subset \widetilde{w}'$ whose initial segment coincides with that of s'. Since $|w'| > |s| + n_0$, we have $v' = s' \cdot y'$, where $|y'| > n_0$.

We first bound the number of possibilities for s'. Since the endpoints of s' are J-close to those of s, for every choice of s there are at most $2V^2$ choices for where s' begins and ends. (The factor of 2 comes from the choice of direction of fellow-traveling.) Turning attention to y', observe that the endpoint of s' is the starting point of y'. Since $|y'| \ge n_0$, Theorem 2.4.(1) and Remark 2.5 imply that there are at most $Be^{b|y'|}$ choices of where y' ends. Thus, for every choice of s, the number of choices for $(s'y')^{\pm 1}$ up to path-homotopy is at most

$$2 \cdot V^2 \cdot Be^{b|y'|} \le 2V^2Be^{b(|w|-|s|+2J)} \le 2V^2Be^{b(\ell-|s|+2J)}. \tag{4.4}$$

Recall that the companion s' is determined by the J–loose cone-piece $s \subset \widetilde{w}$. There are $|w| \le \ell$ possible choices of where the projection of s begins in the closed-geodesic w. There are also choices for the length $|s|$, constrained by the inequalities $d \le |s| < |w'| \le \ell$. Summing over these possible choices, we conclude that for every conjugacy class $[g]$, the number of possibilities for $[g']$ such that the J–loose cone-piece s has diameter at least d is bounded above by

$$\sum_{|s|=d}^{\ell} \ell \cdot 2V^2Be^{b(\ell-|s|+2J)} < \sum_{|s|=d}^{\infty} \ell \cdot 2V^2Be^{b(\ell-|s|+2J)} = \frac{2V^2B\ell\, e^{b(\ell-d+2J))}}{1-e^{-b}}. \tag{4.5}$$

By increasing the constant B if needed, we may ensure that the above exponential bound subsumes the linear bound in the nongeneric warm-up case.

Next, recall that $\mathcal{G}(\ell)$ denotes the set of nontrivial conjugacy classes of translation length at most ℓ. By Theorem 2.4 (2), the number of such conjugacy classes satisfies

$$|\mathcal{G}(\ell)| \ge \frac{A}{\ell} e^{b\ell}.$$

Dividing the previous two equations produces a bound on the fraction of conjugacy classes $[g']$ whose representative closed-geodesic w' has a J–loose cone-piece of diameter at least d with w. This fraction is at most

$$\frac{2V^2B\ell e^{b(\ell-d+2J)}}{1-e^{-b}} \cdot \frac{\ell}{A} e^{-b\ell} \;=\; e^{-bd} \cdot \ell^2 \cdot \frac{2V^2Be^{2bJ}}{A(1-e^{-b})} \;=\; e^{-bd} \cdot \ell^2\, M.$$

Here, $M = \frac{2V^2Be^{2bJ}}{A(1-e^{-b})}$ is a constant that depends only on X and J.

The proof of conclusion (2) is very similar. Suppose that $s \subset \widetilde{w}$ is a J–loose cone-piece between $\widetilde{w}$ and $h\widetilde{w}$, of diameter $|s| \geq d > 2J$. Since d is large, Definition 4.8 applies. Let $s' \subset h\widetilde{w}$ be the companion of s, and let $u = s \cap h^{-1}(s')$ be the overlap. By hypothesis, we have $|u| \leq R$. Let $(v')^{\pm 1} = s' \smallsetminus h(u) = s' \smallsetminus h(s)$ be the portion of s' that is not in the image of the overlap. Then s and v' project to disjoint portions of the closed-geodesic $w \subset X$, and the sign ± 1 is chosen so that s and v' are oriented consistently along w. Thus there is a fundamental domain for w of the form $y_1 v' y_2 s$.

Observe that $|y_1 v' y_2| \leq \ell - |s|$, and the subsegment v' of length $|v'| \geq |s'| - R \geq |s| - 2J - R$ is entirely determined by s and a bounded amount of extra data. Thus, as in equation (4.4), we compute that for every choice of s, the number of choices for $y_1 v' y_2$ up to path-homotopy is at most

$$2\ell\, V^2 B e^{b(\ell - 2|s| + 2J + R)}.$$

Compared to (4.4), the extra factor of ℓ comes from the choice of where in the fundamental domain the subword v' occurs, and the extra constant R in the exponent comes because we have removed the overlap from s'. Since there are at most $Be^{b|s|}$ choices for s, we conclude that the number of possibilities for w is bounded above by

$$2\ell\, V^2 B^2 e^{b(\ell - |s| + 2J + R)} \leq 2\ell\, V^2 B^2 e^{b(\ell - d + 2J + R)}.$$

The remainder of the proof of Lemma 4.11 (1), comparing the above upper bound to the lower bound $\mathcal{G}(\ell) \geq (A/\ell)e^{b\ell}$, goes through verbatim. We conclude that the fraction of conjugacy classes $[g]$ whose representative closed-geodesic has a J–loose cone-piece with itself, with diameter $\geq d$ and overlap of length $\leq R$, is bounded by

$$2\ell\, V^2 B^2 e^{b(\ell - d + 2J + R)} \cdot \frac{\ell}{A} e^{-b\ell} = e^{-bd} \cdot \ell^2 \cdot \frac{2V^2 B^2 e^{b(2J+R)}}{A}.$$

Setting $M_0 = 2V^2 B^2 e^{b(2J+R)}/A$ completes the proof. □

Lemma 4.12. *Suppose X is compact and $\widetilde{X}$ is δ-hyperbolic. Choose a nontrivial element $h \in \pi_1 X$. Suppose that yx is a path from $p \in \widetilde{X}$ to $hp \in \widetilde{X}$. Suppose that each of x and y is a geodesic, and that $|x| \leq J$ for some $J \geq 0$. Define L to be the maximal length of a terminal segment of y that is a 2δ fellow-traveler with an initial segment of hy.*

Let $(yx)^\infty$ denote the bi-infinite path $\cdots h^{-1}(yx)h^0(yx)h^1(yx)h^2(yx)\cdots$. Then $(yx)^\infty$ is an η-quasigeodesic, with η depending only on J, δ, and L.

Proof. If $|y|$ is small, there are only finitely many possibilities for yx. The bi-infinite path $(yx)^\infty$ lies at bounded distance from some quasi-axis of h. So, take the worst case scenario for the quasigeodesic constant η.

If $|y|$ is large, we have large geodesic subpaths of $(yx)h(y)$. By hypothesis, we have a bound L on the length of backtracking in $(yx)h(y)$. Now, apply a "local to global" principle for quasigeodesics. See, e.g., Cannon [Can84, Thm 4]. □

We can now use Lemmas 4.10, 4.11, and 4.12 to show that large J–loose pieces between a closed-geodesic and itself are exponentially rare.

Lemma 4.13. *Suppose X is compact and $\widetilde{X}$ is hyperbolic. Fix a constant $J \in \mathbb{N}$. Then there is a constant $M = M(X, J)$ such that for all sufficiently large ℓ and for all $d \in [2J + 1, \ell/3]$, the following holds. Among all conjugacy classes $[g]$ of* *length at most ℓ, the proportion whose representative closed-geodesic has a J–loose cone-piece with itself, with diameter $\geq d$, is less than $M\ell^2 e^{-bd}$.*

Proof. Consider a closed-geodesic w representing the conjugacy class $[g]$. As a warm-up, we dismiss the (nongeneric) possibility where $|w| \leq 2d$. Since $d \leq \ell/3$, it follows that $|w| \leq 2\ell/3$. By Theorem 2.4.(2), the total number of conjugacy classes of length at most $2\ell/3$ is at most $Be^{2b\ell/3}$, whereas the total number of length at most ℓ is at least $Ae^{b\ell}/\ell$. Thus, the probability that $|w|$ is at most 2/3 of the allowed length is

$$\frac{\mathcal{G}(2\ell/3)}{\mathcal{G}(\ell)} \leq \frac{Be^{2b\ell/3}}{Ae^{b\ell}/\ell} = \frac{B\ell}{A} e^{-b\ell/3} \leq \frac{B}{A} \ell\, e^{-bd}.$$

Thus the conclusion of the lemma holds for $M_1 = B/A$. Note that this warm-up case did not use any hypotheses about pieces between w and itself.

Now, suppose that a closed-geodesic w representing the conjugacy class $[g]$ has a J–loose cone-piece with itself. Following Definition 4.4, let $s \subset \widetilde{w}$ be a preimage of the piece in w, where the endpoints of s lie at distance J from $h\widetilde{w}$. Following Definition 4.8, let $s' \subset h\widetilde{w}$ be the companion of s, and let $u = s \cap h^{-1}s'$ be the h-overlap in s.

Let $R = R(J, \delta) \geq 10J$ be the constant of Lemma 4.10. By Lemma 4.11.(2), pieces with an overlap of diameter $|u| \leq R$ occur with probability at most $M_2\ell^2 e^{-bd}$, where $M_2 = M_2(X, J) = M_0(X, J, R(J, \delta))$ in the notation of Lemma 4.11. Thus, we may suppose that $|u| > R$. By Lemma 4.10, h preserves the orientation on s. We now consider two cases.

Case 1 ($|s| \leq \frac{1}{2}|w|$). Without loss of generality, suppose that the overlap $u \subset s$ occurs at the beginning of u. Then there is a subgeodesic $y_1y_2y_3 \subset \widetilde{w}$, where $s = y_2y_3$ and $s' = h(y_1y_2)$, so that $y_2 = s \cap h^{-1}s'$ is the h-overlap. Then $|y_2| \geq R \geq 10J$, where the second inequality comes from Lemma 4.10. Consequently,

$$|y_1| = |y_1y_2| - |y_2| \leq |y_1y_2| - 10J \leq (|y_2y_3| + 2J) - 10J \leq \tfrac{1}{2}|w| - 8J,$$

and we can conclude that $|y_1y_2y_3| < |w|$. Thus we may choose a fundamental domain for w of the form $y_1y_2y_3z$.

Let x_1 denote a geodesic from the endpoint of y_1 to the start of $h(y_1)$, and let x_3 denote a geodesic from the endpoint of y_3 to the start of $h(y_3)$. Then $|x_1|, |x_3| \leq J$. Since y_1y_2 is a geodesic and $h(y_1)$ fellow-travels with y_2, we see that in the path $y_1x_1h(y_1)$ there is a uniform upper bound on the amount of (2δ)–fellow-traveling between the terminal subpath of y_1 and the initial subpath of $h(y_1)$. Thus, in the notation of Lemma 4.12, we conclude that $(x_1y_1)^\infty$ is an η-quasigeodesic for some $\eta = \eta(\delta, J) > 0$. Likewise, $(y_3x_3)^\infty$ is an η-quasigeodesic. These quasigeodesics fellow-travel, since they are periodically joined by translates of y_2. See Figure 4.

Let $\delta' = \delta'(\delta, J)$ be a uniform constant such that $\eta(\delta, J)$-quasigeodesic quadrilaterals in $\widetilde{X}$ must be δ'-thin. Then it follows that $(x_1y_1)^\infty$ and $(y_3x_3)^\infty$ are δ'–fellow-travelers. Since the endpoint of y_3 is δ'-close to $(x_1y_1)^\infty$, we have an η-quasigeodesic triangle with sides consisting of a subpath of $(x_1y_1)^\infty$, a copy of (y_2y_3), and a geodesic segment of length at most δ' (shown dashed in Figure 4). Since the dashed segment is δ'-short, all of y_2y_3 is $2\delta'$-close to $(x_1y_1)^\infty$. Since $|x_1| \leq J$, all of y_2y_3 is determined by y_1 and a universally bounded amount of data. Since $|y_2y_3| \geq d$, the same argument as in Lemma 4.11 (2), shows that the

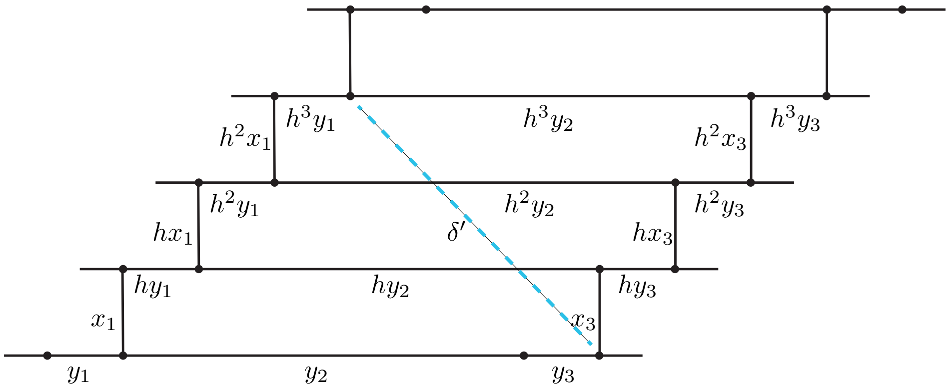

FIGURE 4. Bottom: y_1y_2 is a geodesic, and hy_1 fellow-travels y_2. Hence there is a bounded amount of fellow-travelling between y_1 and hy_1. This implies $(y_1x_1)^\infty$ is a quasigeodesic, and similarly for $(y_3x_3)^\infty$. The two quasigeodesics must δ'–fellow-travel, hence there is a geodesic of length δ' from the endpoint of y_3 to some point on $(y_1x_1)^\infty$.

probability of a J–loose cone-piece with this configuration is at most $M_3\ell^2e^{-bd}$, for a constant $M_3 = M_3(X, J)$.

Case 2 ($|s| > \frac{1}{2}|w|$). By the warm-up argument at the beginning of the proof, we may assume $d \le \frac{1}{2}|w| < |s|$. As in Case 1, we may assume without loss of generality that the overlap u occurs at the beginning of s. Let $\mathring{s}$ be the terminal subgeodesic of s, of length $|\mathring{s}| = d$. By Lemma 4.9, there is a subgeodesic $\mathring{s}' \subset s'$ whose endpoints are $\mathring{J}$-close to those of $\mathring{s}$, for $\mathring{J} = J + 2\delta$. Let $\mathring{u} = \mathring{s} \cap h^{-1}(\mathring{s}')$. We think of $\mathring{s}$ as a subpiece and $\mathring{u}$ as a suboverlap.

If $|\mathring{u}| \le R = R(J, \delta)$, observe that $\mathring{s}$ and $\mathring{v}' = \mathring{s}' \smallsetminus h(\mathring{u})$ project to disjoint portions of the closed-geodesic w, and that $\mathring{v}'$ is determined by $\mathring{s}$ and a universally bounded amount of extra data. Thus, by Lemma 4.11 (2), the probability of a J–loose cone-piece containing this configuration is at most $M_4\ell^2e^{-bd}$, where $M_4 = M_4(X, J) = M_0(X, \mathring{J}, R(J, \delta))$ in the notation of Lemma 4.11.

If $|\mathring{u}| \ge R = R(J, \delta)$, we employ the argument of Case 1, with $\mathring{J}$ in place of J, to show that the probability of a J–loose cone-piece with this configuration is at most $M_5\ell^2e^{-bd}$, for a constant $M_5 = M_5(X, J)$.

To complete the proof, recall the constants $M_1, \dots, M_5$ defined above, so that the probability of a J–loose cone-piece of each type is bounded by $M_j\ell^2e^{-bd}$ for the appropriate M_j. Setting $M = \max_j M_j$ completes the proof. $\square$

Proposition 4.14. *Suppose X is compact and $\widetilde{X}$ is hyperbolic. Fix constants $J \in \mathbb{N}$ and $\phi \in (0, 1)$ and $C > 2/b(1 - \phi)$. Then, for all sufficiently large ℓ and for all $d \in [C \log \ell,\ \ell/3]$, the following holds.*

Suppose that conjugacy classes $[g], [g']$ are chosen at random from among those of length at most ℓ. Then there is an $e^{-bd\phi}$ upper bound on the probability that there is a J–loose piece of diameter at least d between the representative closed-geodesics.

Similarly, for a randomly chosen conjugacy class $[g]$ of length at most ℓ, there is an $e^{-bd\phi}$ upper bound on the probability that there is a J–loose piece of diameter at least d between the representative closed-geodesic and itself.

Proof. Let $M = M(X, J)$ be the larger of the two constants in Lemma 4.11 (1) and Lemma 4.13. Combining the two lemmas, we see that a J–loose cone-piece of diameter at least d in the closed-geodesic representing $[g]$ occurs with probability at most $M\ell^2 \, e^{-bd}$.

Now, let $C > \frac{2}{b(1-\phi)}$ be as in the statement of the lemma. Then, choosing ℓ large ensures that the additive difference $\big(C \log \ell - \frac{2 \log \ell}{b(1-\phi)}\big)$ is as large as we like. In particular, for $\ell \gg 0$ and $d \geq C \log \ell$, we have

$$d \geq C \log \ell > \frac{2 \log \ell}{b(1-\phi)} + \frac{\log M}{b(1-\phi)} = \frac{\log(M\ell^2)}{b(1-\phi)}.$$

After exponentiating and rearranging terms, we obtain

$$M\ell^2 < e^{bd(1-\phi)}, \qquad \text{hence} \qquad M\ell^2 e^{-bd} < e^{-b\phi d}. \qquad \square$$

4.4. **Controlling pieces among many conjugacy classes.** We can now apply Proposition 4.14 to bound the probability of a large J–loose cone-piece between a pair of conjugacy classes that are sampled from a large collection, where “large” is defined as a fraction of the systole. See Proposition 4.17. We can then combine this result with Proposition 4.7 to prove the main theorem.

Lemma 4.15. *Fix constants $q \in \big(0, \frac{1}{6}\big]$ and $J \in \mathbb{N}$. Let $k \leq e^{c\ell}$, where $c < qb$ and b is the growth exponent of $\widetilde{X}$. Select conjugacy classes $[g_1], \ldots, [g_k]$ at random from among those of length at most ℓ, with each $[g_i]$ represented by a closed-geodesic w_i.*

Then, with overwhelming probability as $\ell \to \infty$, all J–loose cone-pieces among the w_i have diameter strictly less than $2q\ell$.

Proof. Since $c < qb$, we may choose a constant $\phi \in (0, 1)$ such that $c < qb\phi^2$. We will consider pieces of diameter at least $d \geq 2q\ell$. Since d is bounded below by a linear function of ℓ, the logarithmic hypothesis on d in Proposition 4.14 is satisfied for $\ell \gg 0$.

The above choices imply that there are $k^2 \leq e^{2c\ell} < e^{2qb\phi^2\ell}$ pairs of indices (i, j). When ℓ is sufficiently large, Proposition 4.14 says that for every pair (i, j), the probability of a J–loose cone-piece of diameter at least d between w_i and w_j is less than $e^{-bd\phi}$. (This includes the case $w_i = w_j$.) Thus the total probability that some pair has a J–loose cone-piece of diameter $\geq d$ is less than

$$k^2 e^{-bd\phi} < e^{2qb\phi^2\ell} e^{-bd\phi} = e^{(2q\phi\ell - d)\, b\phi}.$$

Recall that $d \geq 2q\ell$ and $\phi \in (0, 1)$. Then, as $\ell \to \infty$, the exponent in the above probability estimate is bounded as follows:

$$(2q\phi\ell - d)\, b\phi \leq (2q\phi\ell - 2q\ell)\, b\phi = (\phi - 1) \cdot 2q\phi \cdot b\ell \longrightarrow -\infty.$$

We conclude that with overwhelming probability as $\ell \to \infty$, there are no J–loose pieces of diameter $d \geq 2q\ell$. $\square$

Lemma 4.16. *Select conjugacy classes $[g_1], \ldots, [g_k]$ at random from among those of length at most ℓ, where $k \leq e^{c\ell}$ for some constant $c < qb$ where $0 < q < 1$. Then, with overwhelming probability as $\ell \to \infty$, we have* $\min\{|g_i| : 1 \leq i \leq k\} \geq (1 - q)\ell$.

Proof. Let w_i be a closed-geodesic representing $[g_i]$. By Theorem 2.4 (2), the conditional probability that $|w_i| \le (1-q)\ell$ given that $|w_i| \le \ell$ is

$$\frac{\mathcal{G}((1-q)\ell)}{\mathcal{G}(\ell)} \le \frac{Be^{b(1-q)\ell}}{Ae^{b\ell}/\ell} = \frac{B}{A}\ell e^{-bq\ell}.$$

Thus the conditional probability that $|w_i| < (1-q)\ell$ for some $1 \le i \le k$ is bounded above by $(k\frac{B\ell}{A})e^{-bq\ell}$. This upper bound approaches zero exponentially quickly when $k \le e^{c\ell}$ and $c < qb$. □

Proposition 4.17. *Fix $J \in \mathbb{N}$ and $\alpha \in \big(0, \frac{2}{5}\big]$. Suppose conjugacy classes $[g_1], \ldots, [g_k]$ are chosen at random from among all those of length at most ℓ. Assume that $k \le e^{c\ell}$ for some constant $c < b\alpha/(\alpha+2)$. Then, with overwhelming probability as $\ell \to \infty$, all J–loose cone-pieces among the closed-geodesics representing $[g_1], \ldots, [g_k]$ have diameter strictly less than $\alpha|g_i|$ for every i.*

Proof. We will use Lemmas 4.15 and 4.16 with $q = \frac{\alpha}{\alpha+2}$. Observe that the hypothesis $\alpha \le \frac{2}{5}$ implies $q \le \frac{1}{6}$, as required for Lemma 4.15.

With this value of q, Lemma 4.15 says that with overwhelming probability as $\ell \to \infty$, all cone-pieces have diameter strictly less than $\frac{2\alpha}{\alpha+2}\ell$. Meanwhile, Lemma 4.16 says that with overwhelming probability as $\ell \to \infty$, $\min_{i=1}^{k}\{|g_i|\} \ge (1-q)\ell = \frac{2}{\alpha+2}\ell$.

Putting together the last two results, we conclude that the diameter of every cone-piece in w_i is less than $\alpha|w_i|$. □

In the same spirit as Lemma 4.16, we have

Lemma 4.18. *Suppose $k \le e^{c\ell}$ for some constant $c < b/2$. Then, with overwhelming probability as $\ell \to \infty$, a set of k randomly chosen conjugacy classes has the property that each one is primitive.*

Proof. By Remark 2.6, the probability that $[g_i]$ is nonprimitive is bounded above by

$$\frac{\ell^2 B}{A}e^{-b\ell/2}.$$

Thus the probability that a k-tuple of conjugacy classes contains a nonprimitive class is at most

$$k\frac{\ell^2 B}{A}e^{-b\ell/2} \le \frac{\ell^2 B}{A}e^{(c-b/2)\ell},$$

which goes to 0 as $\ell \to \infty$ because $(c - b/2) < 0$. □

We can now restate and prove Theorem 1.1.

Theorem 1.1. *Let $G = \pi_1 X$, where X is a compact nonpositively curved cube complex, and suppose that G is hyperbolic. Let b be the growth exponent of G with respect to $\widetilde{X}$, and let a be the maximal growth exponent of a stabilizer of an essential hyperplane of $\widetilde{X}$. Let $k \le e^{c\ell}$, where*

$$c < \min\left\{\frac{(b-a)}{20}, \frac{b}{41}\right\}.$$

Then with overwhelming probability as $\ell \to \infty$, for any set of conjugacy classes $[g_1], \dots, [g_k]$ with each $|g_i| \le \ell$, the group $\overline{G} = G/\langle\langle g_1, \dots, g_k \rangle\rangle$ is hyperbolic and is the fundamental group of a compact, nonpositively curved cube complex.

Proof. We may assume that G is nonelementary; otherwise, $a = b = 0$ and the theorem holds vacuously. We begin by replacing X with a closely related cube complex that satisfies Convention 4.3. First, we perform a cubical subdivision of X, while retaining the original metric. Then every conjugacy class $[g] \subset G$ can be assigned a closed-geodesic representative $w \to X$. Second, if some hyperplane of X is inessential, we replace X by its essential core, as in [CS11, Proposition 3.5], which has the same growth exponents a and b.

For every relator $[g_i]$ in the statement of the theorem, let $w_i \to X$ be the chosen closed-geodesic. Let $\widetilde{w}_i \subset \widetilde{X}$ be a geodesic axis that covers w_i, stabilized by $g_i \in [g_i]$. For each $\widetilde{w}_i$, let $\widetilde{Y}_i = \text{hull}(\widetilde{w}_i) \subset \widetilde{X}$, whose quotient $Y_i = \langle g_i \rangle \backslash \widetilde{Y}_i$ admits a local isometry into X. By Lemma 4.2, $\widetilde{Y}_i$ lies in a uniform neighborhood of $\widetilde{w}_i$, hence its quotient Y_i is compact and a quasicircle. By construction, every hyperplane of $\widetilde{Y}_i = \text{hull}(\widetilde{w}_i)$ cuts $\widetilde{w}_i$. This gives us a cubical presentation $X^* = \langle X \mid Y_1, \dots, Y_k \rangle$ such that $\pi_1 X^* = \overline{G}$.

Define $J = J(\widetilde{X})$ as in Lemma 4.5. Then every diameter $\ge d$ cone-piece of X^* between Y_i and Y_j corresponds to a J–loose cone-piece between w_i and w_j, of diameter $\ge d$. Similarly, every diameter $\ge d$ wall-piece of X^* in w_i corresponds to a J–loose wall-piece, also of diameter $\ge d$.

The hypotheses on c allow us to choose a constant $\alpha < \frac{1}{20}$, such that $c < \alpha(b-a)$ and $c < \frac{b\alpha}{\alpha+2}$. Indeed, $c < \frac{1}{20}(b-a)$ and $c < \frac{b/20}{1/20+2} = \frac{b}{20} \cdot \frac{20}{41}$. Then Propositions 4.7 and 4.17 ensure that with overwhelming probability as $\ell \to \infty$, the J–loose (wall or cone) pieces in every w_i have diameter strictly less than $\alpha|w_i| = \alpha\|Y_i\|$. Thus, by Lemma 4.5, the wall-pieces and cone-pieces in a relator Y_i of X^* also have diameter strictly less than $\alpha\|Y_i\|$. Thus X^* satisfies the $C'(\alpha)$ small-cancellation condition. Since $\alpha < \frac{1}{14}$, Lemma 3.4 ensures that $\pi_1 X^*$ is hyperbolic.

Next, we check the hypotheses of Theorem 3.5. We have verified that with overwhelming probability, X^* is $C'(\frac{1}{20})$ and that every Y_i is a compact quasicircle that deformation retracts to a closed-geodesic w_i. By Lemma 4.5 (3), every hyperplane $U \subset Y_i$ has a carrier $N(U)$ of diameter strictly less than $\alpha\|Y_i\|$, which implies that $N(U)$ is embedded. Since U must cut the closed-geodesic w_i, we also conclude that $Y_i \smallsetminus U$ is contractible. Finally, Lemma 4.18 implies that with overwhelming probability, every w_i is primitive. Therefore, Theorem 3.5 ensures that $\pi_1 X^*$ acts freely and cocompactly on the CAT(0) cube complex dual to the wallspace on $\widetilde{X^*}$. □

5. Generalization to other metric spaces

In this section, we prove Theorem 1.3, which generalizes Theorem 1.1 to the setting of groups acting on other, noncubical metric spaces. The idea is to prove a probabilistic statement for quotients $\overline{G} = G/\langle\langle g_1, \dots, g_k \rangle\rangle$ where the relators g_i are sampled from all short conjugacy classes in a G-action on some metric space Υ. As discussed in Section 1, there are many situations (for instance, hyperbolic manifolds) where the most natural geometry associated to a group G is carried by a metric space that is not cube complex. The results of this section enable us to draw conclusions using the growth of Υ rather than the growth of a cube complex.

5.1. **Cube-free definitions and results.**

Convention 5.1. The following assumptions and terminology hold throughout this section. Let G be a nonelementary, torsion-free group acting properly and cocompactly on a δ-hyperbolic geodesic metric space Υ. Let $H_1, \ldots, H_m$ be a collection of infinite index quasiconvex subgroups of G, which will remain fixed for the rest of this section. (In the main case of interest, the H_i are hyperplane stabilizers of some action of G on a CAT(0) cube complex $\widetilde{X}$.) Fix a basepoint $\upsilon \in \Upsilon$.

We assume that every nontrivial element $g \in G$ stabilizes a geodesic axis $\widetilde{\gamma} \subset \widetilde{\Upsilon}$. For every conjugacy class $[g]$, we choose a representative closed-geodesic $\gamma \subset G\backslash\Upsilon$. Then every $g \in [g]$ stabilizes some preimage $\widetilde{\gamma}$ of γ.

Definition 5.2 (Loose pieces in Υ)**.** Fix a constant $J > 0$. Consider bi-infinite geodesics $\widetilde{\gamma}, \widetilde{\gamma}' \subset \Upsilon$ that do not share an endpoint in $\partial\Upsilon$. Observe that $\widetilde{\gamma} \cap \mathcal{N}_J(\widetilde{\gamma}')$ is a closed set because $\mathcal{N}_J(\widetilde{\gamma}')$ is a closed neighborhood, and is bounded because the geodesics do not share an endpoint. Thus $\widetilde{\gamma} \cap \mathcal{N}_J(\widetilde{\gamma}')$ is compact. A *J–loose cone-piece between $\widetilde{\gamma}$ and $\widetilde{\gamma}'$* is the maximal geodesic segment $s \subset \widetilde{\gamma}$ whose endpoints lie in $\mathcal{N}_J(\widetilde{\gamma}')$. The *companion of s* is the maximal segment $s' \subset \widetilde{\gamma}'$ whose endpoints are at distance J from the corresponding endpoints of s.

Now, let γ, γ' be closed-geodesics in $G\backslash\Upsilon$. Then a *J–loose cone-piece* between γ and γ' is the projection of a J–loose cone-piece between arbitrary preimages $\widetilde{\gamma}$ and $\widetilde{\gamma}'$, excluding the case where $\widetilde{\gamma} = \widetilde{\gamma}'$.

A *J–loose wall-piece in $\widetilde{\gamma}$* is a maximal geodesic segment $s \subset \widetilde{\gamma}$ whose endpoints are contained in $\mathcal{N}_J(gH_ip)$ for one of the chosen subgroups H_i and for some $g \in G$. A *J–loose wall-piece in a closed-geodesic* γ is the projection of a J–loose wall-piece in $\widetilde{\gamma}$.

Definition 5.3 (Cube-free $C'(\alpha)$ presentations)**.** Let Υ, δ, υ, G, and $H_1, \ldots, H_m$ be as in Convention 5.1. Let κ_j be the quasiconvexity constant of the orbit $H_j\upsilon$, and let $\kappa = \max_j\{\kappa_j\} + 2\delta$.

Let $g_1, \ldots, g_k$ be infinite-order elements of G. For each g_i, let $\gamma_i \to G\backslash\Upsilon$ be the chosen closed-geodesic representing $[g_i]$. Since each g_i stabilizes an axis $\widetilde{\gamma}_i \subset \Upsilon$, we have $|\gamma_i| = |g_i|_\Upsilon = [\![g_i]\!]_\Upsilon$ in Definition 2.1.

The presentation $\langle G : H_1, \ldots, H_m \mid g_1, \ldots, g_k\rangle$ is called $C'(\alpha)$ *with respect to* $(\Upsilon, \upsilon, \gamma_1, \ldots, \gamma_k)$ if

(1) For every g_i, g_j, the diameter of any 2δ–loose cone-piece between γ_i and γ_j is less than $\alpha[\![g_i]\!]_\Upsilon$.
(2) For every g_i, any κ–loose wall-piece in γ_i has diameter less than $\alpha[\![g_i]\!]_\Upsilon$.

The results of Section 4 have the following generalization to this context.

Proposition 5.4. *Let G, Υ, and $\{H_i\}$ be as in Definition 5.3. Let b be the growth exponent of G acting on Υ. Consider conjugacy classes $[g_1], \ldots, [g_k]$ in G, chosen uniformly from all conjugacy classes of Υ-length at most ℓ. Suppose $k \leq e^{c\ell}$ for some constant $c < b\alpha/(\alpha+2)$. Then, with overwhelming probability as $\ell \to \infty$, property (1) of Definition 5.3 holds.*

Proposition 5.5. *Let G, Υ, and $\{H_i\}$ be as in Definition 5.3. Let b be the growth exponent of G, and let a be an upper bound on the growth exponents of the H_j. Note that all lengths and growths are measured with respect to the action on Υ.*

Consider conjugacy classes $[g_1], \ldots, [g_k]$ *in* G*, chosen uniformly from all conjugacy classes of* Υ*-length at most* ℓ*. Suppose* $k \leq e^{c\ell}$ *for some* $c < \alpha(b-a)$*. Then, with overwhelming probability as* $\ell \to \infty$*, property* (2) *of Definition* 5.3 *holds.*

Sketch of proof. The proofs of Propositions 5.4 and 5.5 are nearly identical to those of Propositions 4.17 and 4.7, respectively. There are two differences in the argument. The primary difference is that Υ is substituted for $\widetilde{X}$, and hyperplane stabilizers are replaced by general quasiconvex subgroups H_i. The requirement that every infinite-order element of $\pi_1 X$ stabilizes a geodesic axis (compare Convention 4.3), which was heavily used in the proofs of Propositions 4.17 and 4.7, is mirrored in our setting by the same requirement in Υ. Note that in both Propositions 4.7 and 4.17, we obtain genericity statements via the counts of Theorem 2.4, which apply perfectly well to the G-action on Υ.

The second difference is more subtle. In the arguments of Section 4, a loose piece $s \subset \widetilde{w}$ is always a combinatorial geodesic segment whose endpoints are at vertices of $\widetilde{X}$. Thus, when we factor a fundamental domain of w as $s \cdot y$, the parts s and y are both combinatorial geodesics, and the number of possibilities for y can be estimated via Theorem 2.4 and Remark 2.5. Meanwhile, in Υ, the set of endpoints of loose pieces $s \subset \widetilde{\gamma}$ might be locally infinite. To enable counting arguments, we make the following adjustment: when we factor a fundamental domain for γ as $s \cdot y$, we perturb both s and y so that they begin and end at points in the G-orbit of the basepoint υ. This adjustment perturbs lengths by a bounded additive error, which becomes absorbed into the multiplicative constants of calculations such as (4.1) and (4.4). Thus perturbing $\widetilde{\gamma}$ to pass through the orbit $G\upsilon$ does not affect the probabilistic conclusions. $\square$

Combining Propositions 5.4 and 5.5 gives Corollary 5.6.

Corollary 5.6. *Let* G*,* Υ*, and* $\{H_i\}$ *be as in Definition* 5.3*. Let* b *be the growth exponent of* G *with respect to* Υ*, and let* a *be an upper bound on the growth exponents of the* H_i*. As above, choose a basepoint* $\upsilon \in \Upsilon$*, and a representative closed-geodesic in* $G \backslash \Upsilon$ *for every conjugacy class in* G*.*

Let $k \leq e^{c\ell}$*, where* $c < \min\{\alpha(b-a), b\alpha/(\alpha+2)\}$*, and consider conjugacy classes* $[g_1], \ldots, [g_k]$ *in* G*, chosen uniformly from all conjugacy classes of* Υ*-length at most* ℓ*. Then, with overwhelming probability as* $\ell \to \infty$*, the presentation* $\langle G : H_1, \ldots, H_m \mid g_1, \ldots, g_k \rangle$ *is* $C'(\alpha)$ *with respect to* $(\Upsilon, \upsilon, \gamma_1, \ldots, \gamma_k)$*.*

5.2. **Translation back to cube complexes.** In Section 4, we used Lemma 4.5 to show that every piece in a cubical presentation is associated with a corresponding J–loose piece, in the sense of Definition 4.4. The following statement is an analogue of Lemma 4.5 that allows us to compare pieces in a cubical presentation $\langle X \mid Y_1, \ldots, Y_k \rangle$ to loose pieces in a G-action on Υ.

Proposition 5.7. *Let* $G = \pi_1 X$*, where* X *is a compact nonpositively curved cube complex whose immersed hyperplanes are essential, and let* $H_1, \ldots, H_m$ *be their fundamental groups. Suppose that* G *acts properly and cocompactly on a* δ*-hyperbolic geodesic metric space* Υ *that admits a* G*-equivariant* λ*–quasiisometry from* $\widetilde{X}$*.*

Let $[g_1], \ldots, [g_k]$ *be conjugacy classes in* G*. For each* i*, let* $\widetilde{Y}_i = \operatorname{hull}(\widetilde{w}_i)$*, where* $\widetilde{w}_i$ *is an axis for* g_i *in* $\widetilde{X}$*. Finally, let* $Y_i = \langle g_i \rangle \backslash \widetilde{Y}_i$*.*

Suppose that, for some $\overline{\lambda} > \lambda$*, the presentation* $\langle G : H_1, \ldots, H_m \mid g_1, \ldots, g_k \rangle$ *is* $C'(\alpha/\overline{\lambda})$ *with respect to* $(\Upsilon, p, \gamma_1, \ldots, \gamma_k)$*. Then there exists a constant*

$M = M(\alpha, \delta, \kappa, \lambda, \overline{\lambda}, X)$ such that whenever $[\![g_i]\!]_\Upsilon \geq M$ for all i, the following holds. The cubical presentation $\langle X \mid Y_1, \ldots, Y_k\rangle$ is $C'(\alpha)$. Moreover, every hyperplane $U \subset Y_i$ has a carrier $N(U)$ of diameter strictly less than $\alpha\|Y_i\|$.

Proof. We begin by recalling the above definition of a λ–quasiisometry. By equation (1.1), there is a (G-equivariant) function $f \colon \widetilde{X} \to \Upsilon$, along with positive constants $\lambda_1, \lambda_2, \epsilon$ such that $\lambda_1\lambda_2 = \lambda$ and

$$\frac{1}{\lambda_1}\mathsf{d}_{\widetilde{X}}(x,y) - \epsilon \;\leq\; \mathsf{d}_\Upsilon(f(x), f(y)) \;\leq\; \lambda_2 \mathsf{d}_{\widetilde{X}}(x,y) + \epsilon. \tag{5.1}$$

We can now relate the systole of Y_i in X to the (stable) translation length of g_i in Υ. By Definitions 3.2 and 2.1, we have

$$\|Y_i\| = |g_i|_X \geq [\![g_i]\!]_X = \lim_{n\to\infty} \frac{\mathsf{d}_{\widetilde{X}}(q, g_i^n q)}{n},$$

for an arbitrary point $q \in \widetilde{X}$. Letting $q' = f(q) \in \Upsilon$ yields

$$\begin{aligned} \lambda_2\|Y_i\| \geq \lambda_2 \lim_{n\to\infty} \frac{\mathsf{d}_{\widetilde{X}}(q, g_i^n q)}{n} &= \lim_{n\to\infty} \frac{\lambda_2 \mathsf{d}_{\widetilde{X}}(q, g_i^n q) + \epsilon}{n} \\ &\geq \lim_{n\to\infty} \frac{\mathsf{d}_\Upsilon(q', g_i^n q')}{n} = [\![g_i]\!]_\Upsilon. \end{aligned} \tag{5.2}$$

There is a constant $\eta = \eta(\delta, \lambda_1, \lambda_2, \epsilon)$ with the following properties. First, for every convex set $S \subset \widetilde{X}$, the image $f(S) \subset \Upsilon$ is η-quasiconvex. Second, for every bi-infinite geodesic $\widetilde{w} \to \widetilde{X}$, the quasigeodesic image $f(\widetilde{w})$ is an η–fellow-traveler with every geodesic that has the same endpoints in $\partial\Upsilon$.

Suppose that there is a diameter d cone-piece in X^* between Y_i and Y_j, where d is very large (the precise criterion will be described below). Following Definition 3.2, this piece is a component of $\widetilde{Y}_i \cap \widetilde{Y}_j$ for appropriate preimages $\widetilde{Y}_i$ and $\widetilde{Y}_j$. Let $x, y \in \widetilde{Y}_i \cap \widetilde{Y}_j$ be points such that $\mathsf{d}_{\widetilde{X}}(x,y) = d$. By Lemma 4.2, there are points $x_i, y_i \in \widetilde{w}_i$ and $x_j, y_j \in \widetilde{w}_j$ that are K-close to x and y, respectively.

The following construction in Υ is illustrated in Figure 5. Let $\widetilde{\gamma}_i, \widetilde{\gamma}_j$ be the representative axes for g_i, g_j, respectively, in Υ. Let $\widetilde{\varpi}_i = f(\widetilde{w}_i)$ be the image of $\widetilde{w}_i$ in Υ, and observe that ϖ_i lies in the η-neighborhood of the axis $\widetilde{\gamma}_i$. Similarly, let $\widetilde{\varpi}_j = f(\widetilde{w}_j)$, and observe that $\widetilde{\varpi}_j$ lies in the η-neighborhood of the axis $\widetilde{\gamma}_j$. Label points $x'_i = f(x_i), y'_i = f(y_i) \in \widetilde{\varpi}_i$ and $x'_j = f(x_j), y'_j = f(y_j) \in \widetilde{\varpi}_j$. Finally, let $x''_i, y''_i \in \widetilde{\gamma}_i$ be points within η of x'_i, y'_i, and similarly define $x''_j, y''_j \in \widetilde{\gamma}_j$.

Observe that the second inequality in equation (5.1) gives

$$\mathsf{d}_\Upsilon(x''_i, x''_j) \leq \mathsf{d}_\Upsilon(x'_i, x'_j) + 2\eta \leq \lambda_2 \mathsf{d}_{\widetilde{X}}(x_i, x_j) + \epsilon + 2\eta \leq \lambda_2 \cdot 2K + \epsilon + 2\eta,$$

and similarly $\mathsf{d}_\Upsilon(y''_i, y''_j) \leq \lambda_2 \cdot 2K + \epsilon + 2\eta$. At the same time, the first inequality in equation (5.1) gives

$$\mathsf{d}_\Upsilon(x''_i, y''_i) \geq \mathsf{d}_\Upsilon(x'_i, y'_i) - 2\eta \geq \frac{1}{\lambda_1}\mathsf{d}_{\widetilde{X}}(x_i, y_i) - \epsilon - 2\eta \geq \frac{d - 2K}{\lambda_1} - \epsilon - 2\eta. \tag{5.3}$$

Thus, when d is sufficiently large, we have $\mathsf{d}_\Upsilon(x''_i, y''_i) > 2\lambda_2 K + \epsilon + 2\eta + 2\delta$.

Next, observe that the geodesic quadrilateral with vertices $x''_j, x''_i, y''_i, y''_j$ is 2δ-thin. Thus every point of $[x''_i, y''_i] \subset \widetilde{\gamma}_i$ must be 2δ-close to some other side of the quadrilateral. Let $x'''_i \in [x''_i, y''_i]$ be the point such that $\mathsf{d}_\Upsilon(x''_i, x'''_i) = 2\lambda_2 K + \epsilon + 2\eta + 2\delta$. Since we have taken d to be large, this point x'''_i exists and is far from y''_i. Then the side that is 2δ-close to x'''_i must be on $\widetilde{\gamma}_j$. The same conclusion

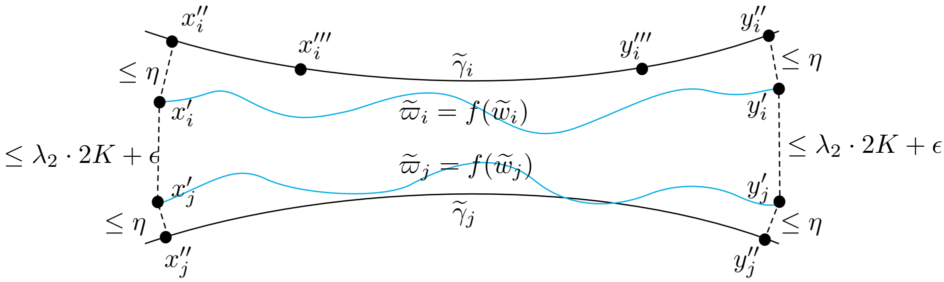

FIGURE 5. The figure lives in Υ. The blue quasigeodesics $\widetilde{\varpi}_i$ and $\widetilde{\varpi}_j$ are the images of axes in $\widetilde{X}$. The black geodesics are axes $\widetilde{\gamma}_i$ and $\widetilde{\gamma}_j$, respectively. Since $\widetilde{\varpi}_i$ must η–fellow-travel with γ_i, there is a point x_i'' that is η-close to x_i', and similarly for the others. The 2δ–loose cone-piece between $\widetilde{\gamma}_i$ and $\widetilde{\gamma}_j$ will contain the segment $[x_i''', y_i''']$.

holds for the point $y_i''' \in [x_i'', y_i'']$ such that $\mathsf{d}_\Upsilon(y_i'', y_i''') = 2\lambda_2 K + \epsilon + 2\eta + 2\delta$. In particular, the entire segment $[x_i''', y_i''']$ lies in $\mathcal{N}_{2\delta}(\widetilde{\gamma}_j)$. Hence there is a 2δ–loose cone-piece in Υ, of diameter at least

$$\mathsf{d}_\Upsilon(x_i''', y_i''') = \mathsf{d}_\Upsilon(x_i'', y_i'') - 2(2\lambda_2 K + \epsilon + 2\eta + 2\delta).$$

Since $\langle G : H_1, \ldots, H_m \mid g_1, \ldots, g_k \rangle$ is $C'(\alpha/\overline{\lambda})$ with respect to $(\Upsilon, p, \gamma_1, \ldots, \gamma_k)$, we get

$$\begin{aligned}
\frac{\alpha}{\overline{\lambda}} [\![g_i]\!]_\Upsilon &> \mathsf{d}_\Upsilon(x_i''', y_i''') \\
&= \mathsf{d}_\Upsilon(x_i'', y_i'') - 2(2\lambda_2 K + \epsilon + 2\eta + 2\delta) \\
&\geq \mathsf{d}_\Upsilon(x_i', y_i') - 2\eta - 2(2\lambda_2 K + \epsilon + 2\eta + 2\delta) \\
&= \mathsf{d}_\Upsilon(x_i', y_i') - (4\lambda_2 K + 6\eta + 4\delta + 2\epsilon) \\
&\geq \frac{1}{\lambda_1} \mathsf{d}_{\widetilde{X}}(x_i, y_i) - \epsilon - (4\lambda_2 K + 6\eta + 4\delta + 2\epsilon) \\
&\geq \frac{1}{\lambda_1} (\mathsf{d}_{\widetilde{X}}(x, y) - 2K) - (4\lambda_2 K + 6\eta + 4\delta + 3\epsilon) \\
&= \frac{1}{\lambda_1} (d - 2K) - (4\lambda_2 K + 6\eta + 4\delta + 3\epsilon).
\end{aligned}$$

Combining the last computation with equation (5.2), we obtain

$$\frac{\lambda_1 \lambda_2}{\overline{\lambda}} \alpha \|Y_i\| \geq \frac{\lambda_1}{\overline{\lambda}} \alpha [\![g_i]\!]_\Upsilon > d - 2K - \lambda_1(4\lambda_2 K + 6\eta + 4\delta + 3\epsilon).$$

Since $\lambda_1 \lambda_2 = \lambda < \overline{\lambda}$, there is a constant M_1 such that when $[\![g_i]\!]_\Upsilon \geq M_1$, we have

$$\alpha \|Y_i\| > \frac{\lambda_1 \lambda_2}{\overline{\lambda}} \alpha \|Y_i\| + 2K + \lambda_1(4\lambda_2 K + 6\eta + 4\delta + 3\epsilon) > d.$$

In other words, the diameter of every cone-piece in Y_i is less than $\alpha\|Y_i\|$. This establishes the cone-piece portion of the claim that $\langle X \mid Y_1, \ldots, Y_k \rangle$ is $C'(\alpha)$.

For the wall-piece portion of the desired conclusion, suppose that $P = \widetilde{Y}_i \cap N(\widetilde{U}_j)$, where $\widetilde{U}_j$ is a hyperplane of $\widetilde{X}$, and suppose that $\operatorname{diam}(P) = d$ is very large. (Note that P may or may not be disjoint from $\widetilde{Y}_i$, hence might not be a wall-piece, according to Definition 3.2. Compare Remark 4.6.) Let $x, y \in P$ be points realizing the diameter. By Lemma 4.2, there are points $x_i, y_i \in \widetilde{w}_i$ that are K-close to x and y, respectively. Similarly, there are points $x_j, y_j \in \widetilde{U}_j$ that are 1-close to x and y, respectively.

Let $\widetilde{\varpi}_i = f(\widetilde{w}_i)$ be the image of $\widetilde{w}_i$ in Υ, and observe that $\widetilde{\varpi}_i$ lies in the η-neighborhood of the axis $\widetilde{\gamma}_i$. Let $x'_i = f(x_i), y'_i = f(y_i)$ be points in $\widetilde{\varpi}_i$, and let $x''_i, y''_i \in \widetilde{\gamma}_i$ be points within η of x'_i, y'_i.

Let $\Theta_j \subset \Upsilon$ be the image of $\widetilde{U}_j$ under the quasiisometry, and let $x'_j = f(x_j), y'_j = f(y_j) \in \Theta_j$ be the images of x_j, y_j. Recall that $\upsilon \in \Upsilon$ is the prechosen basepoint, and that each orbit $H_j\upsilon$ is κ_j-quasiconvex. Note that $\Theta_j \in \mathcal{N}_\psi(gH_j\upsilon)$ for some $g \in G$, where ψ depends on $X, \lambda_1, \lambda_2, \epsilon, \kappa_j$. Let $x''_j, y''_j \in gH_j\upsilon$ be points that are ψ-close to x'_j, y'_j, respectively.

First, observe that

$$\begin{aligned}
\mathsf{d}_\Upsilon(x''_i, x''_j) &\le \mathsf{d}_\Upsilon(x''_i, x'_i) + \mathsf{d}_\Upsilon(x'_i, x'_j) + \mathsf{d}_\Upsilon(x'_j, x''_j) \\
&\le \eta + (\lambda_2 \mathsf{d}_{\widetilde{X}}(x_i, x_j) + \epsilon) + \psi \\
&\le \eta + (\lambda_2(K+1) + \epsilon) + \psi.
\end{aligned}$$

The same estimate holds for $\mathsf{d}_\Upsilon(y''_i, y''_j)$.

The geodesic quadrilateral with vertices $x''_j, x''_i, y''_i, y''_j$ is 2δ-thin. Thus every point of $[x''_i, y''_i] \subset \widetilde{\gamma}_i$ must be 2δ-close to some other side of the quadrilateral. Let $x'''_i \in [x''_i, y''_i]$ be the point such that $\mathsf{d}_\Upsilon(x''_i, x'''_i) = \eta + \lambda_2(K+1) + \epsilon + \psi + 2\delta$. (As with cone-pieces, such a point x'''_i exists whenever d is sufficiently large.) Then the side that is 2δ-close to x'''_i must be the opposite side, namely $[x''_j, y''_j]$. The same conclusion holds for the point $y'''_i \in [x''_i, y''_i]$ such that $\mathsf{d}_\Upsilon(y''_i, y'''_i) = \eta + \lambda_2(K + 1) + \epsilon + \psi + 2\delta$. Thus

$$x'''_i, y'''_i \;\in\; \widetilde{\gamma}_i \cap \mathcal{N}_{2\delta}([x''_j, y''_j]) \;\subset\; \widetilde{\gamma}_i \cap \mathcal{N}_\kappa(gH_j\upsilon).$$

The last containment uses the property $\kappa \ge 2\delta + \kappa_j$, where $gH_j\upsilon$ is κ_j-quasiconvex.

Since $\langle G : H_1, \dots, H_m \mid g_1, \dots, g_k\rangle$ is $C'(\alpha/\overline{\lambda})$ with respect to $(\Upsilon, p, \gamma_1, \dots, \gamma_k)$, we get

$$\begin{aligned}
\frac{\alpha}{\overline{\lambda}} [\![g_i]\!]_\Upsilon &> \mathsf{d}_\Upsilon(x'''_i, y'''_i) \\
&= \mathsf{d}_\Upsilon(x''_i, y''_i) - 2(\eta + \lambda_2(K+1) + \epsilon + \psi + 2\delta) \\
&\ge \mathsf{d}_\Upsilon(x'_i, y'_i) - 2(\eta + \lambda_2(K+1) + \epsilon + \psi + 2\delta) - 2\eta \\
&= \mathsf{d}_\Upsilon(x'_i, y'_i) - 2(2\eta + \lambda_2(K+1) + \epsilon + \psi + \delta) \\
&\ge \frac{1}{\lambda_1} \mathsf{d}_{\widetilde{X}}(x_i, y_i) - \epsilon - 2(2\eta + \lambda_2(K+1) + \epsilon + \psi + \delta) \\
&\ge \frac{1}{\lambda_1} (\mathsf{d}_{\widetilde{X}}(x, y) - 2K) - 2(2\eta + \lambda_2(K+1) + 2\epsilon + \psi + \delta) \\
&= \frac{1}{\lambda_1} (d - 2K) - 2(2\eta + \lambda_2(K+1) + 2\epsilon + \psi + \delta).
\end{aligned}$$

Combining the last computation with equation (5.2), we obtain

$$\frac{\lambda_1\lambda_2}{\overline{\lambda}} \alpha \|Y_i\| \ge \frac{\lambda_1}{\overline{\lambda}} \alpha [\![g_i]\!]_\Upsilon > d \; - 2K - 2\lambda_1(2\eta + \lambda_2(K+1) + 2\epsilon + \psi + \delta).$$

Since $\lambda_1\lambda_2 = \lambda < \overline{\lambda}$, there is a constant M_2 such that when $[\![g_i]\!]_\Upsilon \geq M_2$, we have

$$\alpha\|Y_i\| > \frac{\lambda_1\lambda_2}{\overline{\lambda}}\,\alpha\|Y_i\| + 2K + 2\lambda_1(2\eta + \lambda_2(K+1) + 2\epsilon + \psi + \delta) > d.$$

Thus $d = \operatorname{diam}(P) < \alpha\|Y_i\|$. This bounds the size of hyperplane carriers in Y_i, as well as wall-pieces involving Y_i. We conclude that when $[\![g_i]\!]_\Upsilon \geq M = \max(M_1, M_2)$ the cubical presentation is $C'(\alpha)$. □

5.3. **Main result.** We can now restate and prove Theorem 1.3. After the proof, we discuss a potential strengthening.

Theorem 1.3. *Let $G = \pi_1 X$, where X is a compact nonpositively curved cube complex, and suppose that G is hyperbolic. Suppose that G also acts properly and cocompactly on a geodesic metric space Υ, where every nontrivial element of G stabilizes a geodesic axis. Suppose that there is a G-equivariant λ–quasiisometry $\widetilde{X} \to \Upsilon$.*

Let b be the growth exponent of G with respect to Υ, and let a be the maximal growth exponent in Υ of a stabilizer of an essential hyperplane of $\widetilde{X}$. Let $k \leq e^{c\ell}$, where

$$c < \min\left\{\frac{(b-a)}{20\lambda}, \frac{b}{40\lambda+1}\right\}.$$

Then with overwhelming probability as $\ell \to \infty$, for any set of conjugacy classes $[g_1], \ldots, [g_k]$ with each $|g_i|_\Upsilon \leq \ell$, the group $\overline{G} = G/\langle\langle g_1, \ldots, g_k\rangle\rangle$ is hyperbolic and is the fundamental group of a compact, nonpositively curved cube complex.

Proof. Observe that our hypotheses on c can be restated as

$$c < (b-a)\frac{1/20}{\lambda} \quad \text{and} \quad c < \frac{b(\frac{1}{20})/\lambda}{(\frac{1}{20})/\lambda + 2}.$$

By continuity, we may choose constants $\alpha < \frac{1}{20}$ and $\overline{\lambda} > \lambda$ such that

$$c < (b-a)\frac{\alpha}{\overline{\lambda}} \quad \text{and} \quad c < \frac{b\alpha/\overline{\lambda}}{\alpha/\overline{\lambda} + 2}.$$

As in the proof of Theorem 1.1, we replace X by its essential core, so that all hyperplanes are essential. This does not affect the multiplicative constant λ in the quasiisometry to Υ. Let $H_1, \ldots, H_m$ be the hyperplane stabilizers in $G = \pi_1 X$. We also subdivide X while retaining the original metric, so that every nontrivial conjugacy class is represented by a closed-geodesic. Both operations preserve the growth exponents a and b. Corollary 5.6 implies that with overwhelming probability as $\ell \to \infty$, the presentation $\langle G : H_1, \ldots, H_m \mid g_1, \ldots, g_k\rangle$ is $C'(\alpha/\overline{\lambda})$ with respect to Υ and any choice of basepoint and axes.

For each i, let $\widetilde{w}_i \subset \widetilde{X}$ be a geodesic axis stabilized by $g_i \in [g_i]$. For each $\widetilde{w}_i$, let $\widetilde{Y}_i = \operatorname{hull}(\widetilde{w}_i) \subset \widetilde{X}$, whose quotient $Y_i = \langle g_i\rangle \backslash \widetilde{Y}_i$ admits a local isometry into X. By Lemma 4.2, $\widetilde{Y}_i$ lies in a uniform neighborhood of $\widetilde{w}_i$, hence its quotient Y_i is compact and a quasicircle. By construction, every hyperplane of $\widetilde{Y}_i$ cuts $\widetilde{w}_i$. This gives us a cubical presentation $X^* = \langle X \mid Y_1, \ldots, Y_k\rangle$ such that $\pi_1 X^* = \overline{G}$.

Now, Proposition 5.7 implies that with overwhelming probability as $\ell \to \infty$, the cubical presentation $X^* = \langle X \mid Y_1, \ldots, Y_k\rangle$ is $C'(\alpha)$ with $\alpha < \frac{1}{20}$. Furthermore, every hyperplane U of every Y_i satisfies $\operatorname{diam} N(U) < \alpha\|Y_i\|$. Since $\alpha < \frac{1}{14}$, Lemma 3.4 ensures that $\pi_1 X^*$ is hyperbolic.

Next, we check the hypotheses of Theorem 3.5. We have verified that with overwhelming probability, X^* is $C'(\frac{1}{20})$ and that every Y_i is compact and deformation retracts to a closed-geodesic. By Proposition 5.7, every hyperplane $U \subset Y_i$ has a carrier $N(U)$ of diameter strictly less than $\alpha\|Y_i\|$, which implies that $N(U)$ is embedded. Since U must cut the closed-geodesic w_i, we also conclude that $Y_i \smallsetminus U$ is contractible. Finally, the same argument as in Lemma 4.18 implies that with overwhelming probability, every g_i is primitive. Therefore, Theorem 3.5 ensures that $\pi_1 X^*$ acts freely and cocompactly on the CAT(0) cube complex dual to the wallspace on $\widetilde{X^*}$. $\square$

Remark 5.8. In the above proof of Theorem 1.3, all of the probabilistic arguments happen inside Corollary 5.6, which combines Propositions 5.4 and 5.5. By contrast, Proposition 5.7 involves a global assumption (a G-equivariant λ–quasiisometry), and the proof works entirely in the language of coarse geometry without invoking any counting or probability. One can envision a strengthening of Proposition 5.7 that tracks the behavior of a typical conjugacy class and a typical piece.

To make this precise, choose a basepoint $x \in \widetilde{X}$ and let $B_n(\widetilde{X}) = \{g \in G : \mathsf{d}_{\widetilde{X}}(x, gx) \leq n\}$. Define $B_n(\Upsilon)$ similarly using the basepoint $\upsilon \in \Upsilon$, and recall from Definition 2.2 that $f_{G,\Upsilon}(n)$ counts the cardinality of $B_n(\Upsilon)$. Define the *mean distortion of* $\widetilde{X}$ *with respect to* Υ to be

$$\tau(\widetilde{X}/\Upsilon) \;=\; \lim_{n\to\infty} \frac{1}{f_{G,\Upsilon}(n)} \sum_{g \in B_n(\Upsilon)} \frac{\mathsf{d}_{\widetilde{X}}(x, gx)}{n} \;=\; \lim_{n\to\infty} \frac{1}{f_{G,\Upsilon}(n)} \sum_{g \in B_n(\Upsilon)} \frac{\mathsf{d}_{\widetilde{X}}(x, gx)}{\mathsf{d}_{\Upsilon}(\upsilon, g\upsilon)}.$$

In words, $\tau(\widetilde{X}/\Upsilon)$ measures the average factor by which an element of G sampled using Υ gets stretched in $\widetilde{X}$. See [CT21, Eqn (1.1) and Thm 1.2] for a proof that the limit exists.

The mean distortion can be bounded as follows. Suppose there is a G-equivariant λ–quasiisometry $f\colon \widetilde{X} \to \Upsilon$, as in equation (5.1), so that $f(x) = \upsilon$. Then

$$\frac{1}{\lambda_1} \leq \frac{1}{\tau(\widetilde{X}/\Upsilon)} \leq \lambda_2.$$

For a chosen $\epsilon > 0$, an element $g \in G$ or a conjugacy class $[g]$ is called Υ-*typical* if

$$\Big(\tau(\widetilde{X}/\Upsilon) - \epsilon\Big)\, [\![g]\!]_{\Upsilon} \leq [\![g]\!]_{X} \leq \Big(\tau(\widetilde{X}/\Upsilon) + \epsilon\Big)\, [\![g]\!]_{\Upsilon}. \tag{5.4}$$

For each $\epsilon > 0$, a conjugacy class sampled uniformly from all those of Υ-length at most ℓ will be Υ-typical with overwhelming probability. Indeed, if Υ is the Cayley graph of G with respect to some generating set, this follows from a large deviation result of Cantrell and Tanaka [CT21, Thm 4.23]. If Υ is itself a CAT(0) cube complex, this follows from a large deviation theorem of Cantrell and Reyes [CR23b, Thm 1.4]. For general Υ, the referee informs us that this can be derived from Cantrell and Reyes [CR23a, Eqn (5.2)].

We now discuss the prospects for removing λ from the statement of Proposition 5.7. The proof begins by observing that every conjugacy class $[g_i]$ satisfies $[\![g_i]\!]_X \geq (\lambda_2)^{-1}[\![g_i]\!]_{\Upsilon}$; see equation (5.2). Since $[g_i]$ is Υ-typical with overwhelming probability, we may replace $(\lambda_2)^{-1}$ by $\big(\tau(\widetilde{X}/\Upsilon) - \epsilon\big)$. Next, the proof of Proposition 5.7 uses the constant λ_1 to pass from the diameter of a piece in X^* to the diameter of a loose piece in Υ; see equation (5.3). If one knew that a cone-piece or wall-piece in X^*, coming from a random presentation as in the statement of Theorem 1.3, also corresponds to an Υ-typical group element of G, one could replace λ_1 by $\big(\tau(\widetilde{X}/\Upsilon)+\epsilon\big)$. The upshot would be that the product $\lambda = \lambda_1\lambda_2$ would be replaced by the quotient $\big(\tau(\widetilde{X}/\Upsilon)+\epsilon\big)/\big(\tau(\widetilde{X}/\Upsilon)-\epsilon\big)$, which approaches 1 as $\epsilon \to 0$.

Whether the pieces coming from a random presentation (sampled using lengths in Υ) can be represented by Υ-typical group elements is an interesting problem. See Question 7.10.

6. Pentagonal surfaces

Throughout this section, we consider the setting where $\Upsilon = \mathbb{H}^2$ is the hyperbolic plane, equipped with a tiling T by regular right-angled pentagons, and $\widetilde{X}$ is the square complex dual to this tiling. We work out the optimal constant λ in a λ–quasiisometry from $\widetilde{X}$ to Υ. This can be considered the first interesting example where Theorem 1.3 applies.

Proposition 6.1. *Let T be the tiling of $\mathbb{H}^2$ by regular right-angled pentagons, with hyperbolic metric $\mathsf{d}_{\mathbb{H}}$. The dual tiling T^*, with five quadrilaterals at every vertex, can be identified with a CAT(0) square complex $\widetilde{X}$, with combinatorial metric $\mathsf{d}_{\widetilde{X}}$. Then the identity map $\mathrm{id}\colon (\widetilde{X}, \mathsf{d}_{\widetilde{X}}) \to (\mathbb{H}^2, \mathsf{d}_{\mathbb{H}})$ is a λ–quasiisometry, where*

$$\lambda = \frac{\operatorname{arccosh}(2K^2+2K+1)}{\operatorname{arccosh}(K+1)} \approx 1.5627 \qquad \textit{for} \qquad K = \cos\left(\tfrac{2\pi}{5}\right).$$

Furthermore, this value of λ is optimal.

As mentioned in Section 1, combining Theorem 1.3 with Proposition 6.1 and the classical work of Huber [Hub59] yields a proof of Corollary 1.4.

The proof of Proposition 6.1 is entirely elementary and largely pictorial; see Figures 6–13. The proof also trades the coarse geometry that has dominated most of this paper for the fine geometry of $\mathbb{H}^2$. We begin with Lemma 6.2, which computes a number of lengths in a single right-angled pentagon using the hyperbolic laws of sines and cosines [Fen89, Rat19]. We then continue with Proposition 6.3, which considers a number of combinatorial possibilities for how a hyperbolic geodesic segment can cross a sequence of several adjacent pentagons.

Lemma 6.2. *Let $P \subset \mathbb{H}^2$ be a regular right-angled pentagon. Label lengths as in Figure 6. Set $K = \cos\left(\frac{2\pi}{5}\right)$. Then the labeled lengths can be expressed as follows:*

$$a = \operatorname{arccosh}\left(\sqrt{K+1}\right) \approx 0.5306. \tag{6.1}$$

$$b = \operatorname{arccosh}\left(\tfrac{1}{\sqrt{1-K}}\right) \approx 0.6269. \tag{6.2}$$

$$c = \operatorname{arccosh}(K+1) \approx 0.7672. \tag{6.3}$$

$$d = \operatorname{arccosh}(2K^2+2K+1) \approx 1.1989. \tag{6.4}$$

$$e = \operatorname{arccosh}\left((2K+1)\sqrt{K+1}\right) \approx 1.2265. \tag{6.5}$$

$$f = \operatorname{arccosh}(4K^2+4K+1) \approx 1.6169. \tag{6.6}$$

Furthermore, the length g of a geodesic segment contained in two adjacent right-angled pentagons, as in Figure 6, satisfies

$$2g = \operatorname{arccosh}(1+2K(8K^2+8K+1)^2) \approx 3.1838. \tag{6.7}$$

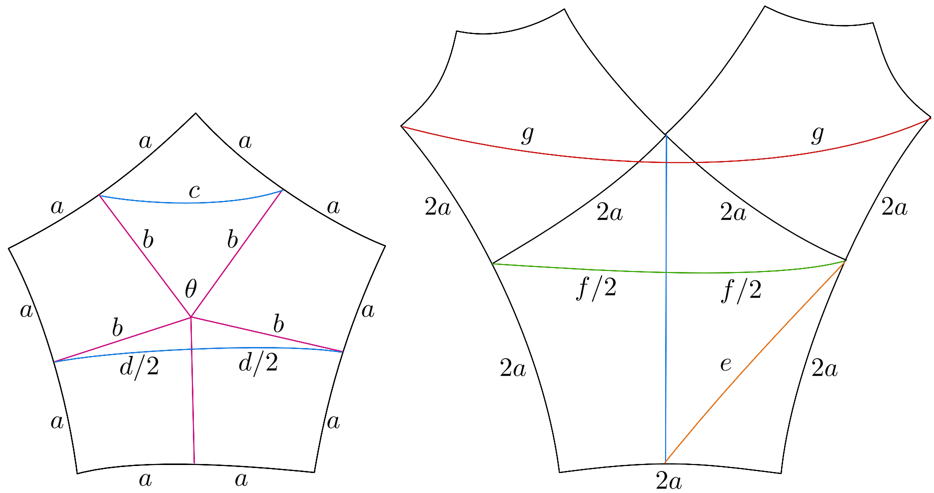

FIGURE 6. Lengths in a right-angled pentagon, as computed in Lemma 6.2.

Proof. In Figure 6, left, the right-angled pentagon P is subdivided into five isometric quadrilaterals, arranged symmetrically about the center point of P. Each quadrilateral Q has two sides of length a, two sides of length b, three right angles, and one angle of $\theta = \frac{2\pi}{5}$. The presence of three right angles makes Q a *Lambert quadrilateral* or *almost rectangular quadrilateral* in the terminology of Ratcliffe [Rat19, Sec 3.5]. By [Rat19, Thm 3.5.9], we have

$$K = \cos\theta = \sinh^2 a,$$

hence $K + 1 = \cosh^2 a$, implying equation (6.1). By [Rat19, Thm 3.5.8], we have

$$\cosh^2 b = \left(\frac{\cos\theta \cdot \cos(\frac{\pi}{2}) + \cosh a}{\sin\theta \cdot \sin(\frac{\pi}{2})} \right)^2 = \frac{\cosh^2 a}{\sin^2\theta} = \frac{1+K}{1-K^2} = \frac{1}{1-K},$$

implying equation (6.2).

The hyperbolic law of cosines [Rat19, Thm 3.5.3] implies the following version of the Pythagorean theorem as a special case. In a hyperbolic right triangle, with legs of length x, y and hypotenuse of length z, the lengths satisfy

$$\cosh x \cdot \cosh y = \cosh z. \tag{6.8}$$

Now, consider the distances between the midpoints of edges in the right-angled pentagon P. By (6.8), the midpoints of two adjacent edges are separated by distance c, where

$$\cosh c = \cosh^2 a = \sinh^2 a + 1 = K + 1,$$

implying equation (6.3). By [Rat19, Thm 3.5.3], the midpoints of two nonadjacent edges are separated by distance d, where

$$\cosh d = \cosh^2 b - \sinh^2 b \cdot \cos(2\theta).$$

Substituting $\cosh^2 b = \frac{1}{1-K}$ and $\sinh^2 b = \frac{K}{1-K}$, as well as $\cos(2\theta) = 2K^2 - 1$ gives

$$\cosh d = \frac{1}{1-K} - \frac{K(2K^2-1)}{1-K} = 2K^2 + 2K + 1,$$

implying equation (6.4).

To compute the lengths e and f, observe that

$$\cosh a = \sqrt{\sinh^2 a + 1} = \sqrt{K+1} \quad \text{and} \quad \cosh(2a) = 2\sinh^2 a + 1 = 2K + 1.$$

Now, (6.8) gives

$$\cosh(e) = \cosh(2a) \cdot \cosh a = (2K+1)\sqrt{K+1},$$

implying equation (6.5). Similarly, (6.8) gives

$$\cosh(f) = \cosh(2a)^2 = (2K+1)^2 = 4K^2 + 4K + 1,$$

implying equation (6.6).

Finally, we use Figure 6 to compute the length g. In that figure, we have a hyperbolic quadrilateral with two right angles at the ends of a side of length $2a$, adjacent sides of length $4a$, and the fourth side of length $2g$. According to first formula in the last block of displayed equations on [Fen89, page 88], we have

$$\cosh(2g) = -\sinh^2(4a) + \cosh^2(4a)\cosh(2a). \tag{6.9}$$

Above, we have already computed that $\cosh(2a) = 2K + 1$, hence

$$\cosh(4a) = 2\cosh^2(2a) - 1 = 8K^2 + 8K + 1.$$

Substituting all this into equation (6.9) gives

$$\begin{aligned}\cosh(2g) &= -\sinh^2(4a) + \cosh^2(4a)\cosh(2a)\\ &= 1 - \cosh^2(4a) + \cosh(2a)\cosh^2(4a)\\ &= 1 + (\cosh(2a) - 1)\cosh^2(4a)\\ &= 1 + 2K(8K^2 + 8K + 1)^2,\end{aligned}$$

implying equation (6.7). □

For the rest of this section, the symbols $a, \ldots, g$ will always denote the constants computed in Lemma 6.2. Now, we can prove the following more global comparison between the cubical and hyperbolic metrics on $\mathbb{H}^2$.

Proposition 6.3. *Let T be the tiling of $\mathbb{H}^2$ by regular right-angled pentagons, with hyperbolic metric $\mathsf{d}_{\mathbb{H}}$. The dual tiling T^*, with five quadrilaterals at every vertex, can be identified with a CAT(0) square complex $\widetilde{X}$, with combinatorial metric $\mathsf{d}_{\widetilde{X}}$. Then there is a constant $\epsilon > 0$ such that for all $x, y \in \widetilde{X}^0$, the distances $\mathsf{d}_{\mathbb{H}}(x,y)$ and $\mathsf{d}_{\widetilde{X}}(x,y)$ can be compared as follows:*

$$c \cdot \mathsf{d}_{\widetilde{X}}(x,y) - \epsilon \;\le\; \mathsf{d}_{\mathbb{H}}(x,y) \;\le\; d \cdot \mathsf{d}_{\widetilde{X}}(x,y) + \epsilon, \tag{6.10}$$

where the constants c, d are as in Lemma 6.2. Furthermore, the multiplicative constants c, d in (6.10) are sharp.

Observe that the constant λ in the statement of Proposition 6.1 is exactly d/c. Thus Proposition 6.3 implies Proposition 6.1.

Proof. We begin by proving the second inequality of (6.10). Let $w \to \widetilde{X}$ be a combinatorial geodesic in $\widetilde{X}$ with endpoints $x, y \in \widetilde{X}^0$. By choosing a sufficiently large additive constant ϵ, we may assume without loss of generality that $|w| = \mathsf{d}_{\widetilde{X}}(x,y) \ge 2$. In the hyperbolic metric on $\mathbb{H}^2$, the combinatorial geodesic w is a concatenation of two or more edges of the dual tiling T^*. By Lemma 6.2, every edge of w has hyperbolic length $2b$.

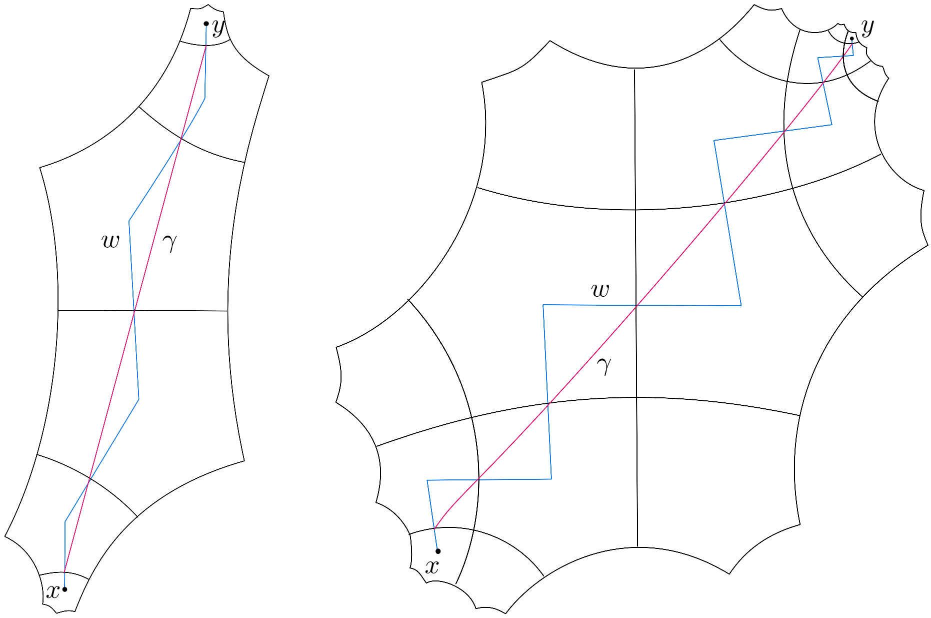

FIGURE 7. The blue path $w \to X$ is a combinatorial geodesic from x to y. The pink path γ is constructed by taking hyperbolic shortcuts between midpoints of consecutive edges of w. Each segment of γ in a pentagon of T has length c or d. The sharpness of the constants c and d in (6.10) is demonstrated by the right and left panels, respectively.

Since edges of T^* meet at angles of $\theta = \frac{2\pi}{5}$ or $2\theta = \frac{4\pi}{5}$, we may homotope w to a shorter piecewise-geodesic path γ by constructing hyperbolic shortcuts between midpoints of consecutive edges. See Figure 7. Each such shortcut replaces two cubical half-edges (of combined cubical length 1) by a hyperbolic segment of length either c or d. The first and last half-edges of w remain as they are, and have hyperbolic length b. Since $c < d$, it follows that

$$\mathsf{d}_{\mathbb{H}^2}(x, y) \;\leq\; b + d(|w| - 1) + b \;\leq\; d|w| + \epsilon \;=\; d \cdot \mathsf{d}_{\widetilde{X}}(x, y) + \epsilon,$$

for an appropriate value of ϵ. Sharpness of the multiplicative constant d holds because one may concatenate arbitrarily many segments of length d in adjacent pentagons to form a hyperbolic geodesic. See the left panel of Figure 7.

By the same token, sharpness of the multiplicative constant c in the first inequality of (6.10) holds because one may concatenate arbitrarily many segments of length c in adjacent pentagons to form a hyperbolic geodesic, as in the right panel of Figure 7.

To prove the first inequality of (6.10), with its optimal multiplicative constant, we make the following definitions. An *altitude* of a right-angled pentagon P is a geodesic segment ℓ from a vertex to the midpoint of the opposite edge. If $s \subset P$ is a hyperbolic geodesic connecting interior points of sides E_1, E_2, the *altitude associated to* s is the unique altitude ℓ with the property that reflection in ℓ interchanges E_1 with E_2. See Figure 8.

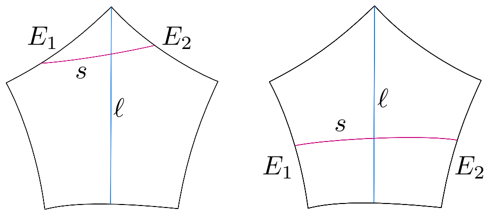

FIGURE 8. The two combinatorial types of segment s cutting through a right-angled pentagon, and the altitude ℓ associated to each s.

Let $\gamma \to \mathbb{H}^2$ be a hyperbolic geodesic with endpoints $x, y \in \widetilde{X}^0$. Then $\mathsf{d}_{\widetilde{X}}(x, y)$ equals the number of hyperplanes that separate x from y. Since the hyperplanes of X are identified with the bi-infinite geodesics containing edges of T, it follows that $\mathsf{d}_{\widetilde{X}}(x, y)$ is the number n of edges of T crossed by γ. After an arbitrarily small perturbation, affecting $\operatorname{len}_{\mathbb{H}}(\gamma)$ by an additive error, we may assume that γ is disjoint from T^0.

In every pentagon P that intersects γ but does not contain the endpoints x, y, draw the altitude associated to $\gamma \cap P$. These $(n-1)$ altitudes partition γ into a concatenation $\gamma_1\gamma_2 \cdots \gamma_n$, where γ_1 is the segment from x to the first altitude; γ_n is the segment from the last altitude to y; and every remaining γ_i connects the altitudes in adjacent right-angled pentagons. To prove (6.10), we will show that the *average* length of a segment γ_i for $0 < i < n$ is at least $c \approx 0.7672$.

There are five combinatorial possibilities for a segment γ_i between two altitudes, corresponding to five types of intersections between γ and two adjacent pentagons. See Figure 9. If ℓ and ℓ' are the altitudes in adjacent pentagons P and P', respectively, the convexity of distance functions implies that the shortest geodesic segment from ℓ to ℓ' meets each of them orthogonally or at an endpoint. Consequently, it is not hard to determine the shortest possible length from ℓ to ℓ'; see Figure 10.

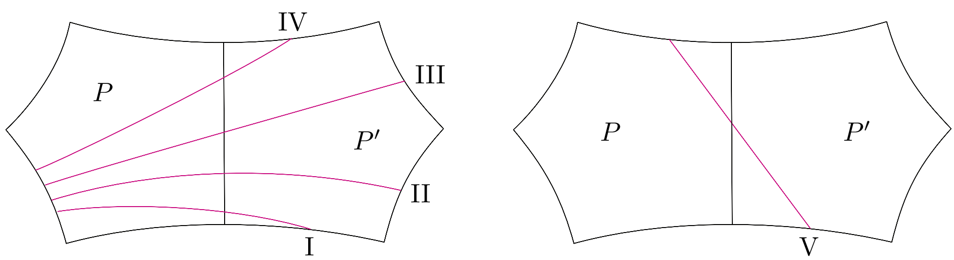

FIGURE 9. The five combinatorial types of (unoriented) intersection between a hyperbolic geodesic and two adjacent pentagons P, P'. In types I–IV, the geodesic γ enters through a side of P that is *not* adjacent to the shared side $P \cap P'$. In type V, the geodesic γ both enters and exits $P \cup P'$ through sides adjacent to the shared side $P \cap P'$.

In type I, Figure 10 shows that the worst-case scenario for $\operatorname{len}(\gamma_i)$ is $a \approx 0.5306$; since $a < c$, this complicated case is analyzed below. In type II, Figure 10 shows that we always have $\operatorname{len}(\gamma_i) \geq 2a > c$. In type III, we always have $\operatorname{len}(\gamma_i) \geq d > c$. In type IV, we always have $\operatorname{len}(\gamma_i) \geq f/2 > c$. Finally, in type V, we always have

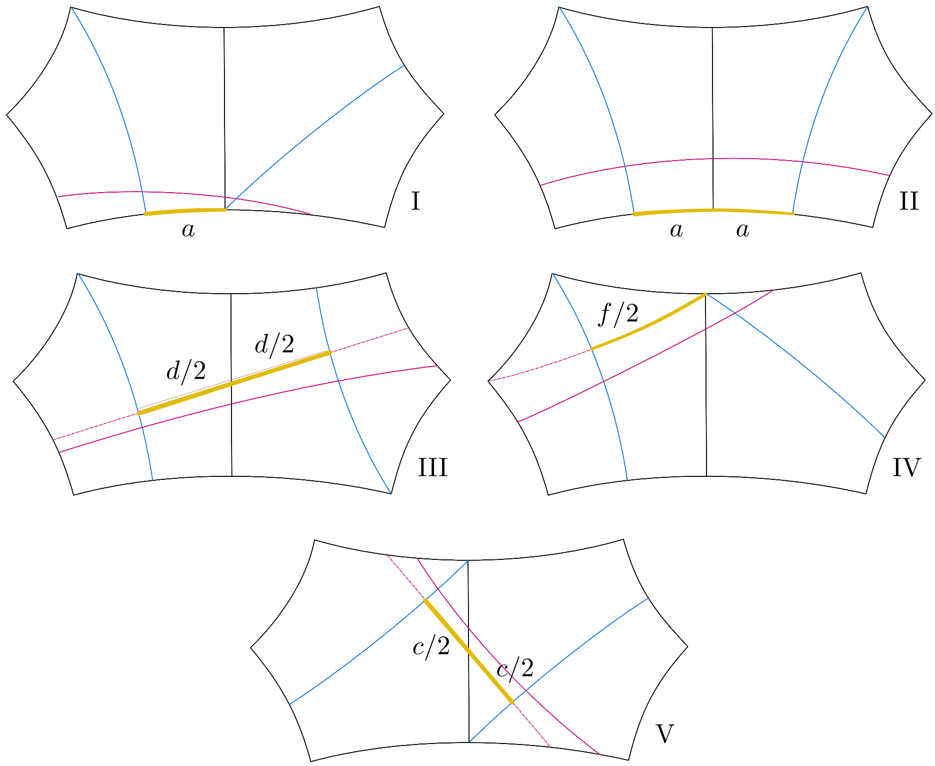

FIGURE 10. For each combinatorial type of intersection between a hyperbolic geodesic and two adjacent pentagons, the highlighted segment shows the infimal length between the altitudes.

$\mathrm{len}(\gamma_i) \geq c$. The comparisons to c come from the numerical values computed in Lemma 6.2. Thus, in every case except type I, we have $\mathrm{len}(\gamma_i) \geq c$.

It remains to analyze segments γ_i of type I. We claim the following:

(1) If γ_i is of type I, and $2 < i < n-1$, then there is an adjacent index $j = i \pm 1$ such that the corresponding segment γ_j is of type II, III, or IV.
(2) If γ_j is of type II or IV, then γ_i is the *only* type-I segment adjacent to γ_j.
(3) For all types of γ_j, we have $\mathrm{len}(\gamma_i) + \mathrm{len}(\gamma_j) \geq \min(3a, g) > 2c$.
(4) If γ_j is of type III, and furthermore γ_j also adjacent to a type-I segment γ_k where $k = i \pm 2$, then $\mathrm{len}(\gamma_i) + \mathrm{len}(\gamma_j) + \mathrm{len}(\gamma_k) \geq 2e > 3c$.

Assuming these claims, we can complete the proof as follows. By claim (1), every segment γ_i of type I (where $3 \leq i \leq n-2$) borrows some of the length from an adjacent segment γ_j of type II, III, or IV. By claim (2), every segment γ_j of type II or IV acts as a "lender" to at most one segment of type I. Every segment γ_j of type III acts as a "lender" to at most two segments of type I. In all cases, claims (3) and (4) say that the average length of γ_j and its adjacent type-I segments is more than c. Putting it all together, we have

$$\mathsf{d}_{\mathbb{H}^2}(x,y) = \mathrm{len}(\gamma) = \sum_{i=1}^{n} \mathrm{len}(\gamma_i) > \sum_{i=3}^{n-2} c = (n-4)c = c \cdot \mathsf{d}_{\widetilde{X}}(x,y) - 4c,$$

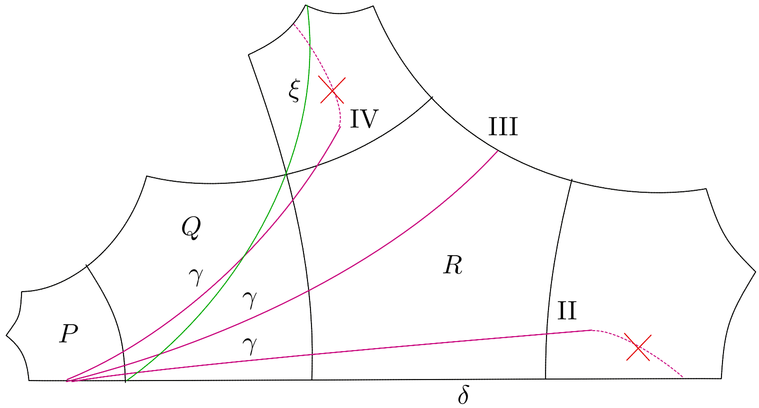

FIGURE 11. Three possibilities for a geodesic γ that intersects $\{P, Q\}$ in a segment of type I. The next segment of γ in pentagons $Q \cup R$ must be of type II, III, or IV.

which completes the proof of equation (6.10).

Figure 11 illustrates the proof of claims (1) and (2). The geodesic γ intersects the left-most pair of pentagons $\{P, Q\}$ in a segment $\gamma_i \subset \gamma$ of type I. We assume without loss of generality that the indices are increasing as γ traverses the figure from left to right. The continuation of γ must exit the central pentagon R through one of the sides marked II, III, or IV, because γ cannot intersect δ twice. This proves claim (1). If the next segment $\gamma_j = \gamma_{i+1}$ is of type II, then the following segment $\gamma_k = \gamma_{i+2}$ cannot be of type I, because otherwise γ would again intersect δ twice. Similarly, if $\gamma_j = \gamma_{i+1}$ is of type IV, then the following segment $\gamma_k = \gamma_{i+2}$ cannot be of type I, because γ cannot intersect the geodesic ξ twice. This proves claim (2).

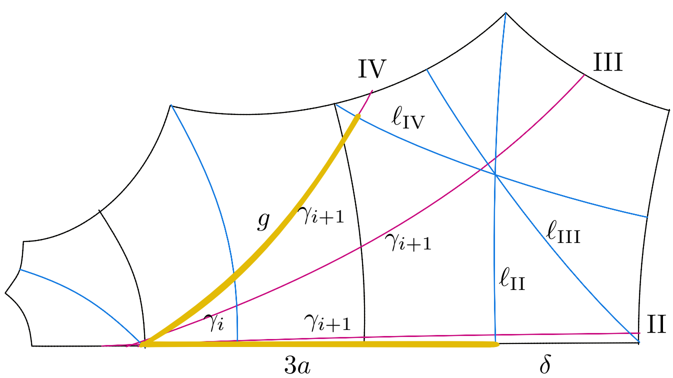

FIGURE 12. The left-most segment γ_i is of type I. The next segment $\gamma_j = \gamma_{i+1}$ is of type II, III, or IV, and terminates at the altitude ℓ_{II}, ℓ_{III}, or ℓ_{IV}, respectively. In type II and type IV, the shortest possible lengths of $\gamma_i \cup \gamma_{i+1}$ are highlighted. In type III, γ_{i+1} must intersect ℓ_{II} or ℓ_{IV}.

For the three combinatorial types of $\gamma_j = \gamma_{i+1}$, Figure 12 illustrates the infimal lengths of $(\gamma_i \cup \gamma_j)$. Let $\ell_{\mathrm{II}}, \ell_{\mathrm{III}}, \ell_{\mathrm{IV}}$ be the terminal altitudes for the three possible types of γ_j. If γ_j is of type II, then the worst-case scenario is when γ fellow-travels δ, hence we obtain $\mathrm{len}(\gamma_i \cup \gamma_j) > 3a > 2c$, where the final inequality uses Lemma 6.2. If γ_j is of type IV, then the worst-case scenario is when $\gamma_i \cup \gamma_j$ starts at the endpoint of an altitude and ends perpendicular to ℓ_{IV}. In this case, we have $\mathrm{len}(\gamma_i \cup \gamma_j) \geq g > 2c$, where the final inequality is by Lemma 6.2. Finally, observe that if γ_j is of type III, then γ_j must intersect either ℓ_{II} or ℓ_{IV}. Thus, by the cases already discussed, we have $\mathrm{len}(\gamma_i \cup \gamma_j) > 2c$. This proves claim (3).

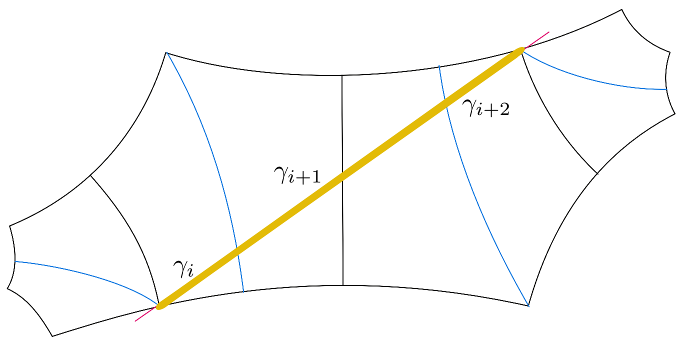

FIGURE 13. If $\gamma_j = \gamma_{i+1}$ is of type III and γ_i, γ_{i+2} are both of type I, the shortest possible configuration is symmetric about the point in the center. In this configuration, each half of $(\gamma_i \cup \gamma_{i+1} \cup \gamma_{i+2})$ is a segment of length e.

Finally, Figure 13 illustrates the proof of claim (4). If $\gamma_j = \gamma_{i+1}$ is of type III and γ_i, γ_{i+2} are both of type I, then the shortest possible length of $(\gamma_i \cup \gamma_j \cup \gamma_k)$ is $2e > 3c$, where the inequality follows from Lemma 6.2. This completes the proof of claims (1)–(4), implying equation (6.10) and Proposition 6.3. $\square$

7. PROBLEMS

This section collects several problems and suggested directions for future research.

Problem 7.1. Determine the optimal density at which random quotients of a surface group G are cubulated. How does the answer depend on the choice of proper metric on G?

Problem 7.2. Given a nonpositively curved cube complex X such that $G = \pi_1 X$ is hyperbolic, find the optimal density such that quotients of G are cubulated. How does the answer depend on X and the growth rate of its hyperplanes?

Problems 7.3–7.5 are stated in order of increasing relatively hyperbolic ambition.

Problem 7.3. Let $G = G_1 * G_2$ be a free product of cubulated groups. Find the optimal density at which random quotients of G are cubulated. Letting $G_i = \pi_1 X_i$, where X_i is a nonpositively curved cube complex, we think of $G = \pi_1 X$, where $X = X_1 \cup A \cup X_2$, where $A \cong [0, n]$ is an arc glued to X_1 and X_2 at its endpoints. Does the optimal density depend on the length of A? See Martin–Steenbock [MS17] and Jankiewicz–Wise [JW22].

Problem 7.4. Let G be a relatively hyperbolic group that is the fundamental group of a compact nonpositively curved cube complex. Show that there is a density at which random quotients of G are cubulated (and relatively hyperbolic).

Problem 7.5. Let G be an acylindrically hyperbolic group that is the fundamental group of a compact nonpositively curved cube complex. Show that there is a density at which random quotients of G are cubulated.

Problem 7.6 has not even been studied for virtually free groups.

Problem 7.6. Generalize Theorem 1.1 to cubulated hyperbolic groups with torsion.

The main challenge is that cubical small-cancellation theory was not described in the case where there is torsion. One could either develop that theory or use another hyperbolic Dehn filling theory with walls (e.g., Osin [Osi07], Groves–Manning [GM08], or Dahmani–Guirardel–Osin [DGO17]) to ensure that the relators embed in the universal cover of the quotient, and subsequently apply the criterion of Theorem 3.5 which can instead be applied to $\widetilde{X^*}$.

Problem 7.7. Let G be a nonelementary relatively hyperbolic group. Prove that there is a density such that all low-density quotients of G are again relatively hyperbolic. Note that the (torsion-free) hyperbolic case was already handled by Ollivier [Oll04].

An unlikely but more fanciful possibility is to use Theorem 1.1 to find new examples of hyperbolic groups that are not cubulated.

Conjecture 7.8. *Let $M = \mathbb{H}^n/\Gamma$ be a closed hyperbolic n-manifold. Assume that either $n \leq 3$ or that M is arithmetic of simplest type. Then, for every $\lambda > 1$, M is homotopy equivalent to a compact nonpositively curved cube complex X such that there is a Γ-equivariant λ–quasiisometry from $\widetilde{X}$ to $\widetilde{M}$.*

Recall that the definition of a λ–quasiisometry appears in equation (1.1).

Conjecture 7.8 is supported by the following intuition. If M is a surface, then there is a plethora of cubulations of M via closed-geodesics. As $\ell \to \infty$, a randomly chosen closed-geodesic γ of length approximately ℓ is nearly equidistributed in M (as well as in T^1M). Therefore, the number of lifts of γ separating a distant pair of points $p, q \in \widetilde{M} = \mathbb{H}^2$ should be nearly proportional to $\mathsf{d}_{\mathbb{H}^2}(p, q)$. The same intuition applies for $n = 3$, where a closed hyperbolic 3-manifold M has a plethora of cubulations via nearly geodesic surfaces that are similarly nearly equidistributed in M. See Kahn and Markovic [KM12].

Similarly, an arithmetic hyperbolic manifold $M = \mathbb{H}^n/\Gamma$ of simplest type always contains totally geodesic (codimension-1) hypersurfaces. Since the commensurator $\mathrm{Comm}(\Gamma)$ is dense in $\mathrm{Isom}(\mathbb{H}^n)$, one may move a single hypersurface by many elements of $\mathrm{Comm}(\Gamma)$, as in [BHW11], to achieve a cubulation of Γ such that, again, $\mathsf{d}_{\mathbb{H}^n}(p, q)$ is nearly proportional to the number of hypersurfaces separating p from q.

Brody and Reyes [BR] have very recently proved Conjecture 7.8, formalizing the intuition in the above discussion.

An ambitious generalization of Conjecture 7.8 asks for arbitrarily homogeneous cubulations of more general groups.

Question 7.9. Let G be a cubulated, hyperbolic group that acts geometrically on a metric space Υ. Is it true that for every $\lambda > 1$, there exists a (proper, cocompact) G-action on a nonpositively curved cube complex $\widetilde{X}$, admitting a G-equivariant λ–quasiisometry $f\colon \widetilde{X} \to \Upsilon$?

Finally, we pose a probabilistic question about pieces that is prompted by Remark 5.8.

Question 7.10. Suppose, as in Theorem 1.3, that $G = \pi_1 X$, where X is a compact nonpositively curved cube complex. Suppose G is hyperbolic, and acts properly and cocompactly on a geodesic metric space Υ. Consider a cubical presentation $X^* = \langle X \mid Y_1, \ldots, Y_k \rangle$.

Let $D = \mathrm{diam}(X)$. For $\epsilon > 0$, a piece P of X^* is called Υ-*typical* if there are points $x, y \in P$ realizing $\mathsf{d}_{\widetilde{X}}(x, y) = \mathrm{diam}(P)$, and an Υ-typical group element $g \in G$ such that $\mathsf{d}_{\widetilde{X}}(gx, y) \leq D$. Recall that Υ-typical group elements are defined in equation (5.4), using an additive constant ϵ.

Suppose that $[g_1], \ldots, [g_k]$ have been sampled uniformly from among all the conjugacy classes satisfying $[\![g]\!]_\Upsilon \leq \ell$, and $X^* = \langle X \mid Y_1, \ldots, Y_k \rangle$ is the cubical presentation associated to $G/\langle\langle g_1, \ldots, g_k \rangle\rangle$. What is the distribution of pieces (above a certain threshold of diameter)? Is it true that with overwhelming probability, all sufficiently large pieces of X^* are Υ-typical?

A positive answer to Question 7.10 would enable one to complete the line of argument outlined in Remark 5.8. We suspect that the answer is "yes" for cone-pieces. We are less confident about wall-pieces, because the distribution of wall-pieces may depend on the Υ-action of hyperplane stabilizers in G.

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